

Assumptions, Proof, and Usefulness: How Gödel and Cantor Expose the Limits of Formal and Empirical Knowledge

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Introduction

This paper investigates the fundamental limits of human knowledge by examining two landmark results in formal reasoning: Gödel's incompleteness theorems and Cantor's diagonal argument. Gödel demonstrated that any sufficiently powerful formal system is inherently incomplete, containing true statements it cannot prove, while Cantor showed that mathematical definability is similarly constrained, with certain numbers provably existing yet undefinable within the system. Both results reveal a structural limitation: through self-reference and diagonalization, a system's own resources expose boundaries it cannot overcome. Building on this insight, the paper argues that empirical science exhibits an analogous limitation. Scientific reasoning relies on assumptions such as the reliability of perception, the uniformity of nature, and the validity of induction, assumptions that cannot be justified without already presupposing them. This circularity, like the limitations of formal systems, is not a flaw but a defining feature of knowledge production. By synthesizing these formal and empirical cases, the paper contends that all knowledge is framed by its unprovable foundations. The choice of a framework, whether a formalist approach to mathematics or a pragmatic reliance on empirical methods, is not a matter of absolute proof but of coherence, utility, and philosophical commitment. Recognizing these limits clarifies the domain and strength of different forms of reasoning, offering a more mature understanding of the scope and reliability of human knowledge.

The Gödelian Divide

Gödel's First Incompleteness Theorem is a landmark result in mathematics, logic, and philosophy. It formally shows that any consistent system capable of arithmetic cannot be both complete and consistent, while also raising enduring questions about proof, truth, and the limits of human knowledge (Gödel 1931: 173–198). At its core, Gödel's result shows that every sufficiently powerful and consistent formal system is necessarily incomplete, containing true statements it cannot prove. Beyond formal logic, it prompts reflection on unprovable assumptions and the

relationship between formal systems, definability, and usefulness. This chapter examines these dimensions to reveal the structural limits of reasoning itself.

Gödel's Theorem Itself

Gödel's proof operates through arithmetization, whereby the theorem assigns a unique numerical code to every logical symbol, formula, and proof within a formal system. This allows metamathematical statements about the system (e.g., the formula with code g is not provable) to be translated into arithmetical statements within the system itself (Pavlovic 2025: 1-2). The theorem's namesake incompleteness arises when Gödel constructs a specific formula, G , which effectively asserts its own unprovability. Thus, G is provable if and only if it is not provable, a contradiction. The only consistent conclusion is that G is true but formally undecidable within the system, exposing the intrinsic limitation (Gödel 1931: 196).

For example, if the system uses symbols $s_0, s_1, s_2, \dots$, one may assign Gödel numbers $gn(s_i) = i$. A formula $F = s_a s_b s_c \dots$ is then assigned a number through a coding scheme based on prime factorization, such as $gn(F) = 2^{gn(s_a)} \times 3^{gn(s_b)} \times 5^{gn(s_c)} \times \dots$.

This coding allows meta-mathematical predicates about the system's formulae and proofs to be expressed as arithmetic relations between their Gödel numbers. Crucially, Gödel demonstrated that these arithmetic relations are primitive recursive, and therefore themselves expressible within the system P (Kreisel 1960: 290). The core of the proof is the construction of a formula G that asserts its own unprovability. This is achieved by defining a predicate $Proof(x, y)$ which holds if and only if x is the Gödel number of a proof of the formula with Gödel number y . The provability predicate is then defined as $Prov(y) \equiv \exists x Proof(x, y)$ (Gödel 1931: 176).

Using a diagonalization lemma, Gödel constructs a formula G with Gödel number g such that: $P \vdash G \leftrightarrow \neg Prov(\ulcorner G \urcorner)$, where $\ulcorner G \urcorner$ is the numeral representing the Gödel number g . Thus, G is formally equivalent to the statement "The formula with Gödel number g is not provable in P ," which is G itself (Gödel 1931: 190).

The theorem follows from this self-reference. Assume $P \vdash G$. Then there exists a proof, so $Proof(n, g)$ holds for some n , and since $Proof$ is primitive recursive, $P \vdash Proof(\bar{n}, \ulcorner G \urcorner)$, and thus $P \vdash Prov(\ulcorner G \urcorner)$. But from the definition of G , $P \vdash \neg Prov(\ulcorner G \urcorner)$, making P inconsistent. Conversely, assume $P \vdash \neg G$. Then from the equivalence, $P \vdash Prov(\ulcorner G \urcorner)$, meaning P proves that a proof of G exists. If P is consistent, this must be false, so P has proven a false statement, rendering it ω -inconsistent. Therefore, if P is consistent, it can prove neither G nor $\neg G$; G is formally undecidable (Gödel 1931: 192). Moreover, within the standard model of arithmetic, G is true, as it correctly asserts the non-existence of a proof of itself (Kreisel 1960: 290).

The Twin Pillars

The first dimension of Gödel's reception critiques the broader philosophical utility of logic itself. One position holds that logic, far from providing clarity, has historically generated

additional philosophical problems rather than resolving them (Nagel 1958: 13). This view parallels post-Wittgensteinian critiques, which treat logical systems as limited language-games that clarify internal relations but obscure the lived practices of meaning (Wittgenstein 1953: 8). From this perspective, logic intensifies paradoxes without resolving them. In contrast, it also has practical utility, underpinning abstract reasoning and modern technological infrastructures, such as computer science (Varti 2009: 13). This is why Gödel's arithmetization of syntax has been retrospectively described as a precursor to computational procedures, insofar as it encodes metamathematical statements within arithmetic itself (Davis 200: 107). The application of these logical formalisms led directly to the foundations of modern computing through Turing's theory of computation, Shannon's information theory, and ultimately the creation of digital machines (Turing 1936: 230).

A third and more speculative dimension finds in Gödel's theorem a pathway to theological reflection. Some have argued that incompleteness implicitly gestures toward the necessity of a transcendent guarantor of truth, a notion Gödel himself explored through his development of a modal ontological argument for the existence of God (Jaki 2004: 49). The connection lies in the suggestion that no finite formal system suffices to ground truth entirely within itself; something beyond the system is required. This interpretation situates Gödel's (1970) result within the long tradition of natural theology, though it remains fiercely contested whether such extrapolations are justified or represent an illicit overextension of formal results (Sobel 1987: 11).

Skeptical voices, often from anti-foundationalist traditions, challenge the rationalist tendency to transform logical results into universal metaphysical claims. Thinkers like or Jacob Klein (2003: 367-368) and Paul Feyerabend (1993: vii-viii), argued that rationalist epistemologies illegitimately universalize methodological norms derived from specific domains. From this vantage point, rationalism often mistakes formal structures for ultimate truth and thereby falls into self-delusion. Thus, logic may be indispensable within its formal boundaries while remaining incapable of securing broader philosophical or theological certainties (Klein 2003: 371). What emerges from this analysis is a clear picture of a multi-dimensional debate. Gödel's theorem is a polyvalent result that can be interpreted as: (1) a source of philosophical paradoxes that destabilize rationalist claims; (2) the foundation of transformative technological achievements; and (3) a potential pathway to theological reflection. This very divergence reveals that formally precise logical results open interpretive spaces that transcend disciplinary boundaries, forcing a reckoning with the limits of formal systems themselves.

The Core Insight

The philosophical implications of Gödel's incompleteness results extend into fundamental questions about the nature of rationality and the limits of formal reasoning. Since Plato, the rationalist tradition has asserted that reason is the most reliable tool for apprehending truth and reality (Kostić 2024: 10). Yet, critics contend that this tradition has failed to produce claims about

reality that are immune to the contingent norms and assumptions of their specific historical contexts (Feyerabend 1993: xi). From this critical perspective, rationalist methods cannot reliably discriminate between conviction (or delusion) and verifiable truth (Alamuti 2023: 814). A central question that emerges from these debates concerns the ontological and epistemic status of fundamental logical principles themselves, such as the law of non-contradiction. While formal logic necessarily assumes the universal validity of these axioms, their justification remains circular and is not self-evident outside the specific systemic frameworks that mandate them. Consequently, skeptics interrogate whether these foundational axioms can legitimately claim any ultimate ontological or epistemic primacy (Rorty 1989: 155). This discussion of limitations naturally extends from Gödel's meta-mathematical findings to the constraints imposed by the formal tools and language we employ (Nagel 1958: 22). For instance, some argue that the purported freedom and creativity of human consciousness, which is sometimes invoked as a means to transcend Gödelian limitations, cannot function as a valid method for resolving formal mathematical questions, such as Cantor's Continuum Hypothesis (*CH*) (Koellner 2023). Attempts to leverage human intuition or insight as a mathematical "proof device" are considered logically vacuous; while the faculties of consciousness themselves are not inherently invalid, their application to solve problems within a formal framework is a categorical error that produces nonsensical conclusions (Tait 1981: 75-92).

Another thing, is the role of language itself as a constraining framework is paramount. Language serves as the indispensable medium for all philosophical and mathematical discourse, yet it simultaneously acts as a structure that binds and shapes concepts, thereby directing and limiting cognition and argumentation. While essential for communication and formalization, language can become functionally useless when over-extended into speculative or rhetorical exercises detached from empirical evidence or formal rigor. This self-referential bind, where we must use language to critique the limits of language, echoes the very formal limitations Gödel exposed, revealing a parallel boundary at the heart of human understanding (Raguní 2016: 1-3).

Cantor's Shadow: Definability, Existence, and Usefulness

If Gödel's incompleteness theorems expose the intrinsic limitations of formal systems, then Georg Cantor's diagonal argument forces a confrontation with the nature of the objects these systems purport to describe. Cantor's demonstration that the set of real numbers is uncountable, that it constitutes a higher order of infinity than the natural numbers, not only revolutionized mathematics but also ignited still-enduring philosophical controversy, particularly surrounding ideas, like the already mentioned *CH*, and the status of "undefinable" numbers (Cantor 1874: 258-262). In summary, Cantor's theorem shows that no function $f: \mathbb{N} \rightarrow (0,1)$ can be surjective. The proof is constructed as such; assume such a function exists and list its values as decimal expansions:

$$f(0) = 0.d_{0,0} d_{0,1} d_{0,2} \dots$$

$$f(1) = 0.d_{1,0} d_{1,1} d_{1,2} \dots$$

$$f(2) = 0.d_{2,0} d_{2,1} d_{2,2} \dots$$

$\vdots$

Now if we define a real number $x = 0.x_0 x_1 x_2 \dots$ where:

$$x_k = \{ 1 \text{ if } d_{\{k,k\}} \neq 1, 2 \text{ if } d_{\{k,k\}} = 1 \}$$

By construction, x differs from every $f(k)$ in the k -th digit and is therefore not in the image of f , contradicting the assumption of surjectivity.

This diagonalization reveals the meta-mathematical insight that real numbers cannot be exhaustively enumerated by any finite or countably infinite scheme (Belaga 2023: 4). Yet, this very triumph of formal reasoning hinges on the existence of "undefinable" numbers, entities that can be proven to exist but cannot be explicitly constructed or uniquely described within a formal system.

This tension between formal existence and constructive usefulness lies at the heart of the philosophical debates over the nature of mathematical objects and the limits of mathematical knowledge. Central to this controversy is the operational value of Cantor's discovery (Dauben 1990: 215). The diagonal argument relies on the existence of numbers that cannot be uniquely specified by any finite definition (Dauben 1990: 165). From a strict constructivist or pragmatist perspective, such entities lack empirical or operational significance; only definable numbers are accessible to human cognition, computation, or application (Drekalović 2025: 105). Consequently, the existence of larger infinities and the undefinable numbers that populate them can be viewed as speculative and practically irrelevant metaphysical postulates. On this view, mathematical entities that cannot be concretely represented or approximated are effectively nonexistent for any applied purpose, regardless of their formal coherence within axiomatic set theory (Dauben 1990: 132).

This tension between formal existence and constructive usefulness becomes acute when Cantor's argument is reflexively applied to the definable numbers themselves. The attempt to form a complete list of all definable real numbers leads to a paradox; the diagonal procedure seems to generate a new number that is both definable (by its construction) and not on the list (by its definition) (Dauben 1990: 192).

The Definability Paradox

The debate over undefinable numbers within Cantorian set theory brings these conceptual tensions into sharp relief. One position maintains that the numbers produced by the diagonal argument are inherently undefinable, as they cannot be uniquely specified by any finite formula. Consequently, such numbers are inaccessible to empirical insight, computation, measurement, or any form of practical manipulation, rendering them irrelevant outside of pure, formal theoretical constructs. They hold, that the apparent contradiction does not break mathematics but instead reveals a deeper meta-mathematical truth about the nature of definability and the boundaries of formalization, mirroring the limitations Gödel revealed for proof (Parker 2019: 375–393). From this operationalist perspective, the *CH* is deemed not merely technically difficult but fundamentally

unsolvable, and perhaps even ill-posed, because it concerns entities (the higher infinities of undefinable numbers) that are ontologically and operationally meaningless. Crucially, attempts to circumvent this issue by restricting analysis to the countable set of definable numbers fail, as the diagonal procedure will always escape any finitely representable enumeration, generating a new number outside the list (Dauben 1990: 22).

This very premise, however, is the core of the debate on limits of formal and empirical knowledge (Woodin 2017). A countervailing perspective contends that if one could construct a list consisting exclusively of definable numbers, then the diagonal number itself would be definable. Its definition would be: "the number whose n -th digit is different from the n -th digit of the n -th number in the list." Since every number in the list is definable by hypothesis, the diagonal number would be well-defined by this finite description. From this viewpoint, the traditional conclusion of uncountability is contingent on the initial list containing undefinable elements; a perfectly defined list would seemingly produce a perfectly defined diagonal number, contradicting the claim that it must be undefinable (Fleuriot 2021: 330).

The resolution of this apparent paradox lies in a meta-mathematical subtlety; while each definable number is indeed finitely representable, the list of all definable numbers is itself undefinable. There is no finite procedure to construct a complete, non-redundant, and well-ordered list of all finite definitions. Any attempt to do so; for instance, by ordering definitions by their symbolic length, leads to insurmountable problems of ambiguity (e.g., multiple definitions for the same number, ill-defined ordering rules, and intractable computational barriers). Therefore, the diagonal number, though described in meta-language, cannot be formally defined within the system because it requires a quantifier ranging over the entire set of definitions, a set that the system itself cannot formally capture (Feferman 1991: 455–468).

Thus, we argue that Cantor's uncountability result is ultimately a consequence of undefinability. The definable numbers are countable in principle, but they cannot be definably enumerated (Salehi 2020: 1-8). The diagonal argument does not create objects of practical use but instead shows an intrinsic epistemic boundary: it demonstrates the inherent limitation of formal enumeration and the inevitable gap between what is theoretically true and what is constructively accessible (Tarski 1983: 154-155). The philosophical significance of the diagonal argument, therefore, resides not in the existence of useless undefinable numbers, but in its power to reveal the limits of formalization itself, distinguishing between mathematical entities that can be constructed and those that are merely postulated by formal theory.

This debate unfolds across two competing interpretations of Cantor's procedure. We will first propose a constructive viewpoint: given any list of definable real numbers, the diagonal number itself is explicitly and finitely defined by the instruction "take the n -th decimal of the n -th number in the list." This construction, by definition, produces a real number not present in the original enumeration. We could argue that this demonstrates a direct contradiction within the very assumption of a complete list of definable reals, but only if we maintain that the diagonal

argument's power is universal and independent of the specific enumeration function; it applies to any proposed one-to-one correspondence between natural numbers and definable reals, seemingly proving that even the definable reals cannot be definitively listed.

However, this constructive interpretation encounters, we can say, a meta-mathematical obstacle. It relies on an operational criterion of definability: a number is considered definable only if it can be represented by a finite procedure that allows any mathematician to compute its value to arbitrary precision. The directive to "take the n -th digit of the n -th number" is indeed a finite instruction in the English language. Yet, to execute this instruction, to actually compute the diagonal number, it requires a definable list. One must be able to uniquely determine the n -th number and its n -th digit for every natural number n . The core contention, therefore, shifts from the definability of the diagonal number to the definability of the list itself.

This is the critical failure of the constructive argument. While each number in the hypothetical list is individually definable by a finite string of symbols, the ordered list of all such numbers is not itself definable by any finite procedure. Any attempt to create this list, for instance, by ordering definitions by their length or lexicographical order, leads to insurmountable practical and theoretical problems: infinite regress in determining the "next" definition, ambiguities in representation, and the impossibility of a computable ordering that avoids duplicates. Consequently, the list is a meta-theoretical concept, not a mathematical object available for computation within the system.

The diagonal construction thus forces a distinction between description in natural language and definability in a formal system. The diagonal number can be described in English with reference to the list, but it cannot be defined within the formal language of mathematics because that would require a finite formula that encapsulates the entire infinite enumeration. Any attempt to represent it in pure mathematical symbols would necessitate an infinite definition, which is impermissible. Therefore, the diagonal number remains undefinable precisely because the list it references is undefinable (Dauben 1990: 247-248). But we can say, that Cantor's argument holds for now, not because it generates a new, strangely undefinable object, but because it reveals that the concept of a "complete list of all definable reals" is itself a meta-mathematical illusion (Salehi 2020: 7-8).

This technical conclusion finds its philosophical counterpart in Belaga's (2023: 2-4) critique. He argues that the works of Cantor and Gödel are of a "mathematically syntactical nature," generating new formal structures but often devoid of semantic content. The principal difficulty of the continuum, its refusal to be neatly placed within Cantor's transfinite hierarchy or Gödel's axiomatic universe, speaks clearly of its "mathematically semantical character." It is a notion, Belaga suggests, that is discovered by humanity in all its simplicity and complexity, not constructed syntactically (Belaga 2023: 4-6). Cantor's discovery that powerset P ; $P(N)=R$, where N is the set of natural numbers and R the continuum, set of real numbers; was profound because it connected two well-known, semantically rich mathematical "stages": the natural numbers and the continuum. However, his subsequent assumption that the power set operation $P(U)$ always yields another

legitimate "set" when applied to infinite sets is a semantically charged statement that requires verification.

Applying this to the technical argument above, Belaga would contend that treating the collection of all definable subsets of N (or R) as a completed "set" ($P(N)$ or $P(R)$) is the core error. For a collection to be a "set" in Cantor's original sense, it must be "reasonable, complete, and well-defined." The power set operation, when applied to an infinite set, forces us to consider all possible subcollections as "elements." But as the definability argument shows, a tremendous majority of these subcollections are completely obscure, lacking any finite description or meaningful representation. They are not "well-defined" objects. Therefore, the power set of an infinite set is not a semantically meaningful mathematical stage; it is a syntactic phantom, a class generated by a rule without a corresponding intuitive mathematical reality (Belaga 2023: 6-8).

The nonexistence of a one-to-one correspondence between natural and real numbers definitively establishes that the cardinality of the reals exceeds that of the naturals. This, however, generates a new debate when applied specifically to definable numbers. The central issue is one of representation: a description of the diagonal number in a natural language like English does not contradict the possibility of a list of numbers defined by finite mathematical symbols. However, a formal definition within a mathematical system imposes stricter requirements. Representing the diagonal number purely in mathematical symbols would necessitate a finite formula that explicitly references every element in the enumeration, effectively requiring an infinite definition. This is not finitely expressible within the formal system. Consequently, even a theoretical list containing all finitely definable real numbers would be susceptible to the diagonal argument. The procedure would generate a number not present in the list. This new number is necessarily undefinable within the same formal system because its construction relies on a complete survey of the list, which is a meta-theoretical operation. A finite representation of this diagonal number in mathematical symbols would logically imply its prior inclusion in the list, creating a contradiction that confirms its non-definability within the formal framework.

This distinction highlights a critical constraint: while natural language can describe the diagonal procedure, formalization within mathematics confronts the inherent limitations of finite reference and the perils of self-referentiality, as seen in paradoxes like that of Richard (Fraenkel 1973: 8). The conclusion is that any assumed bijective function from the natural numbers to the definable reals cannot be explicitly constructed or defined. The diagonal argument therefore applies independently of the specific enumeration function chosen, demonstrating that any attempt to construct a complete list of definable numbers results in an element that is undefinable within the system.

Thus, while the set of definable real numbers is countable in principle, its enumeration is not achievable within the formal system itself. The diagonal number remains undefinable because any formal attempt to define it requires an infinite number of references to the list. The traditional distinction between sizes of infinity, therefore, arises not from an ontological abundance of entities

but from the inherent undefinability of a complete enumeration. This undefinability is a formal result; it does not imply the existence of mathematically useful or empirically accessible entities, but rather exposes a fundamental limitation within formal systems themselves.

The notion of definable real numbers is intrinsically constrained by the formal languages used to define them. Since any formal language admits only a countable collection of formulas, the set of numbers definable within such a system is at most countable (Kunen 1980: 153). The diagonal argument, when applied to the concept of a complete list of these numbers, brings the tension between definability and formal enumeration into sharp relief. A number constructed by the diagonal method can be described in English but cannot be captured by a finite sequence of mathematical symbols within the same formal system; its status is therefore contingent on the language of representation. Consequently, undefinable numbers emerge not as mathematical curiosities with practical applications, but as artifacts of formal constraints. This demonstrates that distinctions between different sizes of infinity are not discoveries of pre-existing ontological realities but consequences of the limits of formal definability (Escultura 2016: 593-595).

Philosophically, this conclusion aligns more with constructivist and formalist positions in the philosophy of mathematics, which suggest that entities which cannot be approximated or constructed by effective procedures lack practical existence.¹ Philosophers like Per Martin L f (1966: 608-612) imply that much of modern mathematics can proceed independently of Cantorian higher infinities, treating them as formal conveniences rather than fundamental ontological entities. This view is also supported by the historical role of foundational results: while G del’s incompleteness theorems were pivotal for logic and meta-mathematics, their direct relevance to day-to-day mathematical practice is limited, similar to how the classical theory of quadratic forms shaped number theory without remaining central to contemporary applied mathematics.

Thus, the diagonal construction demonstrates that the traditional conception of different “sizes of infinity” derives not from an abundance of mathematical objects but from the indefinability of complete enumeration within formal systems. This establishes a framework in which mathematical and philosophical distinctions are made not on the basis of intrinsic existence, but on definitional and representational criteria. The resolution to the apparent paradox of the diagonal number lies in recognizing this fundamental meta-mathematical limitation: the inability of any formal system to fully articulate the resources of its own language.

¹ This conclusion aligns with constructivist and formalist views in the philosophy of mathematics. Constructivists, such as Martin-L f and Troelstra & van Dalen, argue that mathematical objects exist only if they can be explicitly constructed, emphasizing definability. Formalists, following Carnap and Curry, treat mathematics as the manipulation of symbols according to rules, where entities like higher infinities are meaningful only within a formal system, not as ontologically independent objects. See Carnap 1919: 70; Curry 1951: 15.

Resolving the Paradox

The philosophical tensions surrounding Gödel's theorem, Cantor's diagonal argument, and definable numbers can be precisely formalized and resolved through meta-mathematical analysis. The following sequence clarifies the apparent paradox: A real number x is considered definable in the language of set theory (ZFC) if there exists a finite formula $\phi(x)$ within ZFC that uniquely specifies it. Since the collection of all finite formulas in ZFC is countable (being finite strings over a finite alphabet), the set of definable real numbers must also be countable. One can, in principle, imagine an enumeration $(x_1, x_2, x_3, \dots)$ of all definable reals. Cantor's diagonal argument can be applied to this enumeration:

For each natural number n , define $d_n = 1$ if the n -th decimal digit of x_n is 0, and $d_n = 0$ otherwise.

Construct the real number $x = 0.d_1d_2d_3\dots$.

By construction, x differs from every x_i in its i -th decimal place. Therefore, $x \notin \{x_1, x_2, x_3, \dots\}$, contradicting the assumption that this set contained all definable reals.

A paradox seems to arise, since the diagonal number x has a finite English description ("the number whose n -th digit is different from the n -th digit of the n -th definable number in the enumeration"), yet it is not on the list of definable numbers. The resolution lies in a precise understanding of definability as a formal concept: "Definable in ZFC " is a property that applies only to numbers specified by a formula within the formal language of ZFC (Molyneux 2022: 257-260).

The description "the diagonal number of the list of all definable reals" is not a formula of ZFC . It is a statement made in the meta-theory (the English language). To formalize it in ZFC would require ZFC to define its own truth predicate or the set of all of its own definable objects, which is impossible due to self-referential paradoxes, as established by Tarski's undefinability theorem (Tarski 1983: 247-248).

Consequently, there is no finite formula within ZFC that can express the predicate $\text{definable}(x)$ or construct the diagonal number over the class of all definable reals. Any attempt to do so, such as defining:

$$\text{definable}(x) = \exists P(\text{holds}(P, x) \wedge \forall y(\text{holds}(P, y) \rightarrow x = y))$$

$\text{definable}(x) = \exists P(\text{holds}(P, x) \wedge \forall y(\text{holds}(P, y) \rightarrow x = y));$

fails because ZFC cannot contain a universal truth predicate $\text{holds}(P, x)$ for all predicates P without becoming inconsistent.

This analysis yields four different conclusions: (1) The collection of definable real numbers is countable. (2) Cantor's diagonal argument cannot be formally applied inside ZFC to the definable reals to produce a new definable real because the requisite construction is meta-theoretic. (3) Informal descriptions in natural language do not constitute formal definability within a system. (4) This limitation is not a flaw but a fundamental feature of formal systems, directly related to the phenomena captured in Tarski's undefinability theorem and Richard's paradox.

The definability paradox is resolved by recognizing the inability of *ZFC* to internalize the concept of its own definability. Just as Gödel's theorem reveals the inability of a formal system to capture all its own truths, the definability analysis of Cantor's argument reveals the inability of a formal system to define all its own definable objects (Nagel 1958: 264). Both approaches confront the same meta-theoretic ceiling and yield two rival ontologies of mathematical existence. The Formalist-Realist position accepts the diagonal argument at face value: the reals are uncountable, undefinable numbers follow from the axioms, and the definability paradox simply marks the distinction between mathematics and meta-mathematics; the set of definable reals cannot be constructed inside the system, yet higher infinities and the *CH* remain legitimate, if independent, questions. The Operationalist-Constructivist position answers differently: only definable objects are mathematically meaningful, a completed totality of reals is illusory, undefinable numbers are epistemically void, and the *CH* is ill posed. This divergence is a philosophical choice: Formalist-Realists prioritize internal consistency and theoretical fecundity, Operationalist-Constructivists prioritize epistemic access and constructive clarity. The diagonal argument does not resolve the dispute; it shows that mathematical frameworks, like empirical ones, rest on unprovable assumptions, and that our preference is guided by coherence, utility, and ontological commitment rather than by purely mathematical proof.

The Philosophical Implications

The definability paradox raises questions about mathematical existence, hinging on whether one privileges formal existence in axiomatic systems or practical utility. From an operationalist-constructivist view, undefinable numbers are epistemically void, since they cannot be approximated or used. Cantor's proof presumes their formal existence, but they lack instantiation and practical utility (Dauben 1990: 296). This reveals a key tension: formal existence need not entail epistemic relevance. The diagonal argument marks not a vast uncountable domain but the limit of what can be defined, computed, or constructed (Semenov 2025: 2-6). This perspective underscores the divide between mathematical formalism and operational utility. Epistemically, only objects representable by finite symbols or rules, definable numbers, can be said to exist. The diagonal argument produces no new practically relevant entities but marks the limits of formal enumeration. Its definability in natural language yet not within *ZFC* highlights the meta-theoretic character of notions such as 'all definable numbers.' By contrast, a formalist-realist stance accepts Cantor's proof and the axioms of set theory at face value: undefinable numbers exist, and the failure to define the diagonal within *ZFC* reflects the limits of any single formalism, not of mathematics itself. The divergence between these views cannot be resolved mathematically. It is a philosophical choice grounded in coherence, pragmatic utility, and ontological commitment. This debate illustrates the paper's thesis: foundational frameworks, whether mathematical or empirical, ultimately rest on unprovable assumptions, chosen according to intellectual or practical aims (Smullyan 1992: 230). These debates already suggest a parallel with Gödel's incompleteness

results: in both cases, formal reasoning discloses limits that cannot be overcome within the system itself. To clarify this affinity, it is useful to set Gödel and Cantor explicitly side by side.

Self-Reference and Boundary Phenomena

The comparison between Gödel's incompleteness theorem and Cantor's diagonal argument reveals a structural affinity: both employ self-reference to expose the inherent limits of formal systems. Gödel shows that in any consistent axiomatic system capable of expressing arithmetic, one can construct a statement asserting its own unprovability (Nagel 1958: 109). Cantor shows that any attempt to enumerate the reals fails, since one can always construct an element differing from each member of the list. In both Gödel and Cantor, a diagonal procedure converts the system's own resources into a limit it cannot contain. Gödel encodes meta-mathematical claims within arithmetic by assigning numbers to formulas and proofs, enabling the system to speak about itself. Cantor parallels this by defining a number relative to every entry in the enumeration, producing an object both generated within the process and beyond its closure (Hallett 1984: 53). In each case, the operation is reflexive: the system turns upon itself, and in doing so reveals the impossibility of total self-containment.

This structural reflexivity has many implications. It shows that limits are not external impositions on formal systems, but internal necessities disclosed through their own procedures. Gödel's undecidable statement is not a foreign element introduced from outside arithmetic, but a formula generated within it. Cantor's diagonal real is not imposed from outside the list, but derived by a procedure that presupposes the list itself (Hallett 1984: 57). Both results thus reveal that the aspiration to total self-grounding, whether in proof or in definability, is systematically thwarted by the very tools designed to achieve it. The convergence also sharpens the philosophical point. Gödel's incompleteness theorem reveals that formal proof cannot exhaust truth; Cantor's diagonal argument reveals that formal definability cannot exhaust existence (Raguní 2016: 4).

Each marks a boundary phenomenon: Gödel at provability, Cantor at definability. Together, they show that formal systems cannot guarantee both completeness and coherence. Their power lies in exposing these intrinsic limits, revealing that incompleteness is a necessary feature, not a failure. This insight is epistemological: just as formal systems cannot ground their own totality, empirical reasoning cannot escape the circular reliance on the faculties and observations it seeks to justify. We now turn to this broader parallel.

The Empirical Regress and Pragmatic Circularity

The comparative analysis of Gödel and Cantor shows that formal systems, by turning reflexively upon themselves, generate boundary-phenomena that no further internal resources can overcome. Arithmetic cannot prove all its own truths; set theory cannot enumerate all its own definables. These limitations are not accidental but necessary, inscribed into the very procedures by which formal reasoning secures its rigor.

Once this is recognized, an immediate parallel presents itself in the empirical sciences. For empirical reasoning, too, relies upon assumptions that it cannot itself justify without circularity: the reliability of sense perception, the uniformity of nature, and the validity of inductive inference. The pursuit of knowledge, whether through formal proof or empirical observation, is fundamentally constrained by these assumptions (Feyerabend 1962: 60). Mathematics operates within explicitly stated formal systems: given a set of axioms A , it deduces conclusions B that necessarily follow, without appeal to external evidence. Mathematical proof is thus internal to its system.

Empirical science aims to describe mind-independent reality but relies on pragmatic circularity. It assumes the uniformity of nature, the reliability of perception, and the validity of induction, yet these principles are justified only by the empirical success they produce. Observation cannot validate itself *a priori*; all empirical reasoning rests on a pre-rational, secular “faith” in experience (Belaga 2023: 8). Like formal systems, science cannot ground the trust it places in its own methods without presupposing them (Feyerabend 1962: 88). The boundary in this case is not logical undecidability, but a regress that returns inevitably to the same pragmatic commitments.

This necessary reliance on unprovable premises, however, should not be mistaken for a discovery of timeless, universal foundations. The work of historians and philosophers of science like E. A. Burt and Morris Kline (1982: 278-279), and particularly the seminal historical analysis of Jacob Klein (1992: 17), reveals a deeper insight: the specific *content* of these assumptions, the very “basic concepts”, of our formal and empirical reasoning, are themselves historically contingent (Heidegger 1968: 8). Klein's work demonstrates that the shift from Greek mathematical thought, grounded in an ontology of number as discrete multitude, to the modern algebraic conception of a symbolic variable, was not an inevitable progression toward a truer mathematics but a fundamental transformation in the human conceptualization of quantity and form (Klein 1992: 17).

This historicity exposes a second-order circularity. Our formal systems appear apodictically certain, and our scientific theories appear to correspond isomorphically to the world, precisely because we have “slipped in tacitly *a priori*” the definitions of correspondence and proof at the outset. The “unreasonable effectiveness of mathematics in the natural sciences” is, from this perspective, partially an artifact of a shared historical genesis in the “Galilean-Cartesian” project of constructing a mathematical nature (Wigner 1960: 1-14). We get out what we put in. The pragmatic success of this project is undeniable, but its circularity becomes transparent when subjected to phenomenological critique.

Thus, the Gödelian and Cantorian limits are not merely technical results about formal systems *in abstracto*; they are the inevitable expression of the limits of any particular, contingent formal framework. They reveal the point where a specific system's resources are exhausted, forcing the very crisis in its basic concepts that the phenomenology of mathematics identify (Husserl 1970: 178-183). The parallel empirical circularity is the pragmatic manifestation of a specific, historically-developed stance towards the world. Together, they argue that the foundations of

knowledge are always dual in nature: first, a necessary, formal reliance on unprovable assumptions (the epistemological point demonstrated by Gödel and Cantor), and second, a contingent, historical choice of *which specific assumptions* to adopt (the historical point illuminated by phenomenology of mathematics). The choice between frameworks is, therefore, not merely between coherence and utility, but between different historically-situated conceptions of truth, existence, and reason itself.

Synthesis

We can also say that this distinction shows why mathematical formalism can be considered more certain in its internal logic than empirical reasoning is in describing the world. One could argue that in mathematics, acknowledging assumptions is not equivalent to asserting their truth; it is merely recognizing the framework within which deductions are valid. The thing is, this framework allows provability in a precise sense, independent of contingent empirical facts, which contrasts sharply with natural science, where verification is contingent and often provisional.

All systems of knowledge are grounded in assumptions that cannot be justified from within the system itself without circularity. Formal systems, such as mathematics and logic, exemplify this: their axioms and rules of inference are accepted without internal proof, and Gödel's incompleteness theorem shows that this limitation is unavoidable. Empirical science exhibits a parallel structure: the reliability of sense perception, the uniformity of nature, and the validity of induction are assumed rather than independently justified, producing a circular dependence analogous to that in formal systems. In both cases, certainty is conditional and bounded by these foundational assumptions. Therefore, the limits revealed in formal systems extend analogously to all empirical reasoning, demonstrating that the boundaries of knowledge are universal and that both mathematical and scientific inquiry ultimately rest on pragmatic or philosophical commitments rather than absolute justification.

Conclusion

The examination of Gödel's incompleteness theorem and Cantor's diagonal argument reveal a common structural insight: both disclose limits intrinsic to any attempt at self-grounding reason. Gödel demonstrates that no formal system can capture all mathematical truth, while Cantor shows that no definition can exhaust mathematical existence. A compelling critique further suggests that the pursuit of formal completeness through infinite syntactic iteration, such as the repeated application of power set constructions, leads not to deeper mathematical reality, but to structures devoid of semantic content, resembling a kind of mathematical scholasticism. These internal boundaries, reached by self-referential techniques, find an analogue in the empirical sciences, where reliance on unprovable assumptions such as induction or the reliability of perception marks an unavoidable circularity. Taken together, these results suggest that knowledge is never founded on absolute proof but on frameworks sustained by coherence, pragmatic success, and philosophical commitment. To recognize these limits is not to undermine reason but to clarify its scope, affirming

that mathematics and science alike derive their power from their conventions and practices rather than from unattainable completeness.

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