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A Mathematical Model of The Best Way to Control the Spread of HIV/AIDS, Taking into Account the Group of People Who are not Affected After Being Exposed to the Disease

Dr. Mireille D. Chastain
Institute of Cellular Biotechnology, Nouvelle Académie des Sciences, France
Prof. Henrik L. Sørgaard
Danish Institute for Clinical Biosciences, Copenhagen, Denmark

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Abstract:
In this study, we created a mathematical model of how HIV/AIDS spreads over time that includes measures to stop transmission after contact. There are six important parts to the model: Susceptible, Exposed within three days, Exposed but Not Infected, Treatment, Pre-AIDS, and AIDS. To find the effective reproduction number, the next-generation operator method was used. The stability analysis showed that the model's disease-free equilibrium point is locally asymptotically stable when the reproduction number is less than one, and it has a unique endemic equilibrium when the reproduction number is greater than one. Both pre-contact preventive measures and anti-retroviral treatment (ART) control measures were added to the optimal control model. To make the best system, Pontryagin's maximum theory was used. Seven ways of controlling things were thought about. "The optimal application of Pre-contact preventive control measures" was found to be the most cost-effective way to keep things under control. People who use both pre-contact preventive and anti-retroviral therapy together were also able to control the virus better and more cheaply than people who used the other two control strategies together. MATLAB was used to simulate how the populations of different classes changed when each of these strategies was used. This was done to see how these control measures affected the virus.
Keywords: Mathematical Modeling; Post-Exposure Prophylaxis; Stability Analysis; Optimal control; Simulations.

1. Introduction

Since it was first found in the 1980s, the Human Immunodeficiency Virus (HIV), which causes AIDS, has continued to be a big public health and economic problem around the world (Marsudi T. 2021; Hattaf K. and Yousfi N. 2021). According to the World Health Organization (2024), AIDS has killed about 40,4 million people around the world as of 2023. Many countries are also seeing a rise in new cases. As of the end of 2024, 40.8 million people were thought to be living with HIV, and 65% of them were in Africa (World Health Organization, 2021). AIDS, which is the later stage of HIV infection, is thought to be a global disease that is spreading quickly. Many diseases affect people all over the world, and Sub-Saharan Africa has the most

of them (Supriya Sarkar et al. 2019). According to genetic research, HIV came from West Africa in the 20th century (Sharp P. M. and Hahn B. H. 2021). In the first few weeks of an infection, which happens in 40–90% of cases, people may have fevers, rashes, headaches, sores in the mouth and genitals, and swollen, painful lymph nodes. Tuberculosis (TB) and other opportunistic infections can also happen to some people. There may also be stomach problems like vomiting and diarrhoea. HIV is mostly passed from person to person through unprotected sexual contact, contaminated blood transfusions, and from mother to child during pregnancy, delivery, or breastfeeding (AIDS gov, 2015). Sharp objects like needles, syringes, blades, and other surgical tools that are dirty can also spread it. When drug users share or reuse dirty needles and syringes, they are increasing their chances of getting HIV. Accidental needle sticks happened when a patient moved without warning, when used needles were handled or thrown away, when they were recapped, when a coworker accidentally stuck someone, and when the needle was taken apart. The virus doesn't live long, though, and doesn't copy itself on surfaces or outside of the body. In Nigeria, there is a lot of worry about how HIV can spread among drug users, especially people who inject drugs (PWIDs). Sharing needles and syringes is one of the main ways that HIV gets spread among PWID. The risk of getting HIV is highest for female sex workers who inject drugs. There is no known way to stop or cure AIDS. Anti-retroviral therapy (ART), on the other hand, makes infected people healthier and lowers their risk of dying as a result. A group of drugs is used in this method, which is often called "highly active antiretroviral therapy" (HAART). People with HIV who can get ART now have life expectancies that are measured in decades in both high-income and low-income countries. In patients who get the best treatment, it may get close to that of the healthy population (Deeks S. G. et al. 2013). ART treatment still has a lot of problems, like not fully restoring health, having side effects, costing a lot of money, and not being able to cure the disease (Silva, C.J., and Torres, D.F., 2017a). When ART use went up, the number of AIDS deaths went down. In 2012, there were about 1.6 million AIDS deaths, down from 2.3 million in 2005 (Silva, C.J., and Torres, D.F. 2017b). For prevention, health professionals say that people should not have any sex without protection and should use condoms during casual sex. They also say that people should get enough checks before getting a blood donation and should not share needles. With the newest information, doctors have come up with Pre-exposure Prophylaxis (PrEP) to protect against getting sick. PrEP is a drug that people who don't have HIV take to lower their chances of getting HIV. Researchers have also made therapeutic drugs that can keep people who are newly exposed to HIV from getting the disease. Post-Exposure Prophylaxis (PEP) stops HIV infection by stopping the virus from spreading and taking hold in the body. It does this by using a group of antiretroviral drugs that stop the virus from growing by blocking different steps in its replication. Taking anti-retroviral drugs within 72 hours of being potentially exposed to HIV is known as PEP (AIDS gov. 2015; Centers for Disease Control and Prevention 2005). This is done to avoid getting HIV.

A lot of authors have used HIV/AIDS transmission as a model. In 2008, Sharomi et al. created a deterministic HIV treatment model that includes both a normal (drug-sensitive) strain and a drug-resistant strain. This model helps us understand how the two strains change over time and

find the best ways to stop the spread of HIV in this situation. A qualitative analysis of their model shows that it has a globally asymptotically stable disease-free equilibrium when a certain epidemiological threshold is less than unity. However, when this threshold goes above unity, the disease will stay in the population.

Gumel wrote an article called "Modelling Transmission Dynamics of HIV/AIDS: Some Results and Challenges" (Gumel Abba B. 2007). In it, he said that antiretroviral drugs (ARVs) must be used for a long time to lower the viral load in HIV-infected people to levels that can't be detected. However, he also said that using these ARVs can cause a resistant strain of HIV to appear and spread. So, he came up with the idea of a tailored (CD-dependent viral-load) treatment. People with an HIV CD4 count of less than 200 cells/ml should be the only ones treated. People who have a very low CD4 count are either pre-AIDS or AIDS and have a high viral load. Many other researchers, including Cheneke K. R. (2023), have also made deterministic models of how HIV/AIDS spreads, but none of them have included the newly exposed Class, which can be treated with PEP to avoid getting the disease. The Centers for Disease Control and Prevention (CDC) of the United States talked about Post-contact Prophylaxis (PEP) and said that it is a way to avoid getting HIV after a possible recent contact (CDC 2015). It means starting ART drugs as soon as possible (within 3 days) after a single high-risk event or after being exposed to HIV. This stops HIV from copying itself and spreading to other parts of the body. This is very helpful for people who got HIV by accident, like health workers and people who are sexually attacked by HIV-positive people. When started before 72 hours of being exposed to the virus, studies show that PEP lowers the chance of getting HIV by more than 80%. It must be taken once or twice a day for 28 days (NIH 2024) Because of these facts, we include the under-3-day HIV Exposed class in our HIV/AIDS transmission model. This way, people who are HIV-exposed can avoid getting infected by getting the right treatments with PEP.

The framework of the work was as follows: an introduction, the creation of a model, an analysis of the model, a numerical simulation of the optimal control model, an analysis of how cost-effective the model is, results, and a conclusion.

2. Model Formulation

Model Assumptions

- ✓ The population enters into the susceptible class at a constant rate Λ_h .
- ✓ The susceptible class becomes infected with HIV through fluid or blood-to-blood contact with the infected individuals.
- ✓ The model does not consider HIV infected persons with no clinical symptoms, migrants with HIV, and those born with the virus or transmissions through sharps.
- ✓ Health workers, sexually abused persons, and those who are accidentally exposed to the virus may seek PEP treatment. Both the pre-AIDS and AIDS classes may receive antiviral treatment to reduce the viral load.

- ✓ Under 3-day HIV-exposed individuals who have judiciously completed their doses of PEP treatment are prevented from getting HIV infection and therefore remain uninfected.
- ✓ Only pre-AIDS, those on Treatment, and the AIDS individuals contribute to the viral transmission. The under-3-day Exposed are people who have had less than 72 hours of effective contact with the virus, are not yet infected, and cannot transmit the disease.

We formulate a deterministic model of HIV/AIDS transmission dynamics. The human population is divided into six compartments based on the epidemiological status of individuals. The compartments are;

Susceptible class (S_h): these are healthy people, not yet exposed to the virus, and are at risk of being exposed to HIV.

Under-3-day Exposed class (E_h): these are individuals who were newly exposed to the virus and are at risk of contracting the disease. The duration for this group is under 72 hours after effective contact with the infected. An under-3-day HIV exposed individual is not yet infected and therefore assumed to be unable to transmit the virus during this period.

Pre-AIDS class (I_h): these are individuals who are infected and infectious but have not progressed to the AIDS class. They do not have AIDS symptoms.

Treatment class (T_h): these are the under-3-day HIV Exposed, pre-AIDS, and AIDS individuals who are receiving treatment. Some of the under-3-day HIV exposed individuals receive PEP treatment to prevent the infection and progress to the uninfected class.

Exposed-Uninfected class (U_h): these are under-3-day HIV exposed individuals who have been satisfactorily free of the virus after receiving PEP treatments.

AIDS class (A_h): these are individuals who have the symptoms of AIDS.

The total human population at time t, represented by $N_h(t)$, is obtained as;

$$N_h(t) = S_h(t) + E_h(t) + I_h(t) + T_h(t) + U_h(t) + A_h(t)$$

The recruitment rate of humans into the susceptible population is denoted as Λ. Susceptible individuals make sufficient contact with the pre-AIDS, infected individuals on treatment, or the AIDS individuals, to progress to the under-3-day exposed window period. At this stage, the infection can be prevented with PEP treatment. They may opt for treatment or progress to the pre-AIDS class. The period of this under-3-day Exposed class is short-lived (within 72 hours of the contact). This class gets exposed to HIV infection at a variable rate of

$$\xi = \frac{\beta((1-\varepsilon)T_h + \eta_1 I_h + \eta_2 A_h)}{N_h},$$ known as the force of infection.

β is the effective contact rate and η_i, $(i = 1,2)$, represents the increase in relative infectiousness rates of individuals in classes I_h and A_h, compared to those in T_h. $(1-\varepsilon)$ represents the proportion of those on treatment that can transmit the virus. Individuals in the pre-AIDS class progress to the AIDS class at the rate l, or seek ATR treatment at the rate ρ. The rest of the parameters are as described in Table 1

Table 1: Descriptions of the parameter

Parameters	Descriptions
μ	Natural death rate
δ	Death due to the virus
α	The rate of progression from the under-3-day Exposed class to the pre-AIDS class
k	The rate at which the under-3-day Exposed individuals receive PEP treatments
m	The rate at which those on treatment return to the pre-AIDS class due to relaxation of their treatment or failure of the drugs.
v	The rate at which individuals in the AIDS class on antiretroviral treatment (ART) progress to the treatment class.
ψ	The rate at which those in the under-3-day exposed class who are on PEP treatment progress to the Exposed-Uninfected class.
ε	The fraction of those on treatment who were exposed but uninfected.
ϖ	Fraction of the under-3-day Exposed individuals that receive PEP treatment.

The transmission dynamics of the virus are represented in the compartmental flow diagram shown in Figure 1

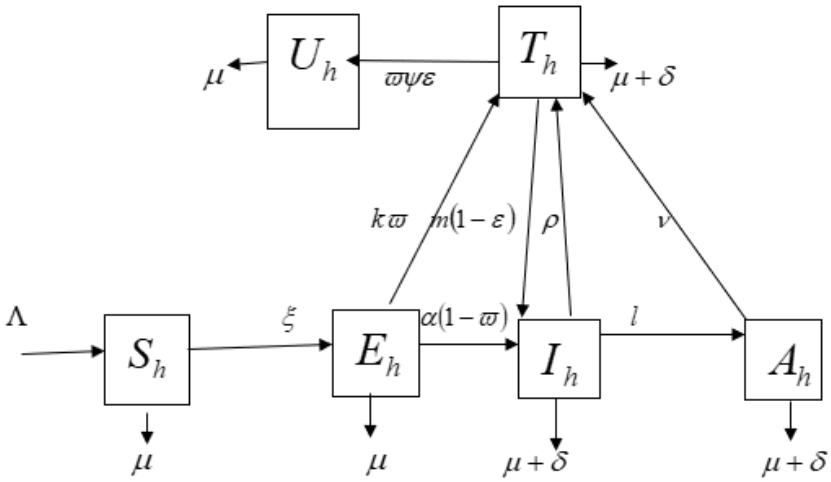

Figure 1: Flow diagram of HIV transmission dynamics

$\varpi\psi\varepsilon$ is the rate at which a fraction of the under-3-day exposed individuals who received PEP progress to the Exposed but uninfected class.

Based on the stated assumptions, parameter descriptions, and flow diagram in Figure1, we present a system of equations governing HIV/AIDS transmission dynamics as (1).

$$\begin{cases} \dfrac{dS_h}{dt} = \Lambda - (\xi + \mu)S_h \\ \dfrac{dE_h}{dt} = \xi S_h - (k\varpi + \alpha(1-\varpi) + \mu)E_h \\ \dfrac{dI_h}{dt} = \alpha(1-\varpi)E_h + m(1-\varepsilon)T_h - (\rho + l + \mu + \delta)I_h \\ \dfrac{dT_h}{dt} = k\varpi E_h + \rho I_h + \nu A_h - (\varpi\psi\varepsilon + m(1-\varepsilon) + \mu + \delta)T_h \\ \dfrac{dU_h}{dt} = \varpi\psi\varepsilon T_h - (\mu)U_h \\ \dfrac{dA_h}{dt} = lI_h - (\nu + \mu + \delta)A_h \end{cases} \quad (1)$$

With the initial conditions

$$S_h(0) > 0,\ E_h(0) > 0,\ I_h(0) > 0, T_h(0) > 0,\ U_h(0) > 0,\ A_h(0) > 0$$

3. Model analysis

3.1 Positivity invariance and boundedness of solutions

Since the system of equations in (1) represents human populations, it is necessary to consider that the associated population sizes cannot be negative. We establish the positive invariant of the epidemiology region Ω given as;

$$\Omega = \left\{ (S_h, E_h, I_h, T_h, U_h, A_h) \in \Re_+^6 : N_h \le \frac{\Lambda}{\mu} \right\}$$

Theorem 1: All the solutions of the system (1) are positive in the region $\Omega \subset \Re_+^6$ for all $t > 0$ provided that the initial conditions are positive.

Proof: Let Ω represent the feasible region of the HIV/AIDS model (1), expressed as $\Omega \subset \Re_+^6$ we show the conditions for the positive invariant of Ω

We obtain the rate of change of the total population $\frac{dN}{dt}$ by adding all the equations in (1) to have;

$$\frac{dN}{dt}=\frac{dS_h}{dt}+\frac{dE_h}{dt}+\frac{dI_h}{dt}+\frac{dT_h}{dt}+\frac{dU_h}{dt}+\frac{dA_h}{dt}$$

$$\Rightarrow \frac{dN}{dt}=\Lambda-\left(S_h+E_h+I_h+T_h+U_h+A_h\right)\mu-\left(I_h+T_h+A_h\right)\delta$$

$$\Rightarrow \frac{dN}{dt}=\Lambda-N_h\mu-\left(I_h+T_h+A_h\right)\delta$$

$$\Rightarrow \frac{dN}{dt}\leq \Lambda-N_h\mu$$

$$\frac{dN}{dt}+N_h\mu\leq\Lambda$$

Multiplying through with the integrating factor $e^{\int \mu dt}=e^{\mu t}$, and integrating gives;

$$Ne^{\mu t}\leq\int \Lambda e^{\mu t}$$

$$Ne^{\mu t}\leq\frac{\Lambda e^{\mu t}}{\mu}+c$$

Solving for the constant c, we apply the initial conditions, when $t=0$, and substitute back to have;

$N(t)\leq\frac{\Lambda}{\mu}+\left(N(0)-\frac{\Lambda}{\mu}\right)e^{-\mu t}$. As $t\to\infty$, we have $N(t)\leq\frac{\Lambda}{\mu}$

For the initial population $0<N(0)\leq\frac{\Lambda}{\mu}$, then $0<N(t)\leq\frac{\Lambda}{\mu}$ for all $t>0$. This implies that the model system (1) is positive invariance and that for all $t>0$, every solution of the model system (1) with the initial condition in the region Ω remains there and is all bounded.

3.2 Disease-free equilibrium state (ε_0)

The HIV/AIDS disease-free equilibrium (DFE) state implies a state where the disease is absent in the population. The DFE is obtained by setting the right-hand side of Equations (1) to zero in the absence of HIV infection. That is;

$$\frac{dS_h}{dt}=\frac{dE_h}{dt}=\frac{dI_h}{dt}=\frac{dT_h}{dt}=\frac{dU_h}{dt}=\frac{dA_h}{dt}=0$$

In the absence of the disease, we have $E_h=I_h=T_h=U_h=A_h=0$. This leads Equations (1) to;

$\frac{dS_h}{dt} = \Lambda - (\xi + \mu)S_h = 0$. The force of infection ξ is also zero at the DFE state, which implies that $S_h = \frac{\Lambda}{\mu}$. Thus, the DFE point is $\varepsilon_0 = \left(\frac{\Lambda}{\mu}, 0,0,0,0,0\right)$

3.3 Effective reproduction number, R_e.

This implies the mean number of new infections generated by a typical infectious individual introduced in a population (Cheneke K.R. *et al.* 2021, Driessche vanden P. 2017). It is a threshold quantity to determine the system's stability. To compute the reproduction number (R_e), we employ the method of the next generation matrix as in (Gumel A.B. 2007, Peter O. 2020). R_e is the spectral radius of the next generation matrix. In this approach, we identify the new infection terms F and the remaining transition terms V, of the infection. The classes where HIV is present are E_h, I_h, T_h and A_h. We can then write (1) as $X = F \times V$ and $V = V^- - V^+$, where $X = (S_h, E_h, I_h, T_h, U_h, A_h)$, V^- is the rate of transfer of the infectious individuals out of each class, and V^+ is the rate of transfer into each class by all other means. Hence, the associated matrices F of the new infection terms and V, the remaining transition terms are given by.

$$F = \begin{bmatrix} \frac{\beta((1-\varepsilon)T_h + \eta_1 I_h + \eta_2 A_h)}{N_h} S_h \\ 0 \\ 0 \\ 0 \end{bmatrix}$$ and

$$V = \begin{bmatrix} (k\varpi + \alpha(1-\varpi) + \mu)E_h \\ (\rho + l + \mu + \delta)I_h - \alpha(1-\varpi)E_h - m(1-\varepsilon)T_h \\ (\varpi\psi\varepsilon + m(1-\varepsilon) + \mu + \delta)T_h - k\varpi E_h - \rho I_h - \nu A_h \\ (\nu + \mu + \delta)A_h - lI_h \end{bmatrix}, \quad (2)$$

We obtain the Jacobian matrices of F and V with respect to the disease-free equilibrium $\varepsilon_0 = \left(\frac{\Lambda}{\mu}, 0,0,0,0,0\right)$. Let the Jacobian matrices of F and V be f and v respectively. Then,

$$f = \begin{bmatrix} 0 & \frac{\beta\eta_1 S_h}{N_h} & \frac{\beta(1-\varepsilon)S_h}{N_h} & \frac{\beta\eta_2 S_h}{N_h} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ and

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$$v=\begin{bmatrix}(k\varpi+\alpha(1-\varpi)+\mu) & 0 & 0 & 0\\ -\alpha(1-\varpi) & (\rho+l+\mu+\delta) & -m(1-\varepsilon) & 0\\ -k\varpi & -\rho & (\varpi\psi\varepsilon+m(1-\varepsilon)+\mu+\delta) & -\nu\\ 0 & -l & 0 & (\nu+\mu+\delta)\end{bmatrix} \quad (3)$$

For computation, let

$$f=\begin{bmatrix}0 & A & B & C\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix} \text{ and } v=\begin{bmatrix}W & 0 & 0 & 0\\ -g & X & -h & 0\\ -i & -j & Y & -k\\ 0 & -l & 0 & Z\end{bmatrix}$$

Computing with Matlab gives the product of f and the inverse of v as the matrix,

$$fv^{-1}=\begin{bmatrix}\frac{-AZ(gY+ih)-B(gjZ+gkl+iXZ)-Cl(gY+ih)}{(-XYZ+jhZ+lkh)} & \frac{-AYZ-B(jZ+kl+iXZ)-CYl}{(-XYZ+jhZ+lkh)} & \frac{-AhZ-BXZ-Chl}{(-XYZ+jhZ+lkh)} & \frac{-Ahk-BXk-C(-XY+hj)}{(-XYZ+jhZ+lkh)}\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$$

For the eigenvalues $\lambda_i,\ \ i=1,2,3,4$, we solve for λ in the characteristic equation

$|fv^{-1}-I\lambda|=0$,

$$\begin{vmatrix}\frac{-AZ(gY+ih)-B(gjZ+gkl+iXZ)-Cl(gY+ih)}{W(-XYZ+jhZ+lkh)}-\lambda & \frac{-AYZ-B(jZ+kl+iXZ)-CYl}{(-XYZ+jhZ+lkh)} & \frac{-AhZ-BXZ-Chl}{(-XYZ+jhZ+lkh)} & \frac{-Ahk-BYk-C(-XY+hj)}{(-XYZ+jhZ+lkh)}\\ 0 & -\lambda & 0 & 0\\ 0 & 0 & -\lambda & 0\\ 0 & 0 & 0 & -\lambda\end{vmatrix}=0$$

to have

$$\lambda_1=\frac{AZ(gY+ih)+B(gjZ+gkl+iXZ)+Cl(gY+ih)}{W(XYZ-jhZ-lkh)},\quad \lambda_2=0,\quad \lambda_3=0,\ \text{and}\ \ \lambda_4=0,$$

Substituting back the parameters in λ_1, to have

$$\lambda_1=\frac{\frac{\beta\eta_1 S_h}{N_h}Z(a(1-\omega)Y+k\omega m(1-\varepsilon))+\frac{\beta(1-\varepsilon)S_h}{N_h}(a(1-\omega)(\rho Z+vl)+kwXZ)+\frac{\beta\eta_2 S_h}{N_h}l(\alpha(1-\omega)Y+k\omega m(1-\varepsilon))}{W(XYZ-\rho m(1-\varepsilon)Z-lvm(1-\varepsilon))} \quad (4)$$

where,

$W=(k\varpi+\alpha(1-\varpi)+\mu),\ X=(\rho+l+\mu+\delta),\ Y=(\varpi\psi\varepsilon+m(1-\varepsilon)+\mu+\delta)\ and\ Z==(\nu+\mu+\delta)$

At DFE, $S_h=\frac{\Lambda}{\mu}$ and the dominant eigenvalue (the spectral radius of the matrix) is λ_1.

Hence, the effective reproduction number is given as;

$$R_e=\frac{\beta\Lambda}{\mu N_h}\left(\frac{\eta_1 Z(\alpha(1-\omega)Y+k\omega m(1-\varepsilon))+(1-\varepsilon)(\alpha(1-\omega)(\rho Z+vl)+kwXZ)+\eta_2 l(\alpha(1-\omega)Y+k\omega m(1-\varepsilon))}{W(XYZ-\rho m(1-\varepsilon)Z-lvm(1-\varepsilon))}\right) \quad (5)$$

To show that $\lambda_1>0$, we simplify the denominator of (5) to clear the negative signs;

That is $W\left(XYZ-\rho m(1-\varepsilon)Z-lvm(1-\varepsilon)\right)=W\left(Z\left(XY-\rho m(1-\varepsilon)\right)-lvm(1-\varepsilon)\right)$

Substitute the values of XY and expand to have

$$W\left(Z\left((\rho+(l+\mu+\delta))(\varpi\psi\varepsilon+m(1-\varepsilon)+\mu+\delta)-\rho m(1-\varepsilon)\right)-lvm(1-\varepsilon)\right)$$
$$=W\left(Z\rho(\psi\varepsilon+\mu+\delta)+Z(l+\mu+\delta)(\varpi\psi\varepsilon+\mu+\delta)+Zm(1-\varepsilon)(l+\mu+\delta)-lvm(1-\varepsilon)\right)$$

Substitute the value of Z in the third term and simplify to have;

$$W\left(\begin{array}{l}Z\rho(\varpi\psi\varepsilon+\mu+\delta)+Z(l+\mu+\delta)(\varpi\psi\varepsilon+\mu+\delta)+vm(1-\varepsilon)l+vm(1-\varepsilon)(\mu+\delta)\\+(\mu+\delta)(m(1-\varepsilon)l)+(\mu+\delta)^2m(1-\varepsilon)-lvm(1-\varepsilon)\end{array}\right)$$
$$=W\left(Z(\varpi\psi\varepsilon+\mu+\delta)(\rho+l+\mu+\delta)+m(1-\varepsilon)(\mu+\delta)(v+l+\mu+\delta)\right)$$
$$=W\left(Z(\varpi\psi\varepsilon+\mu+\delta)X+m(1-\varepsilon)(\mu+\delta)(l+Z)\right)$$

Giving;

$$\lambda_1=\frac{\frac{\beta\eta_1 S_h}{N_h}Z\left(\alpha(1-\omega)Y+k\omega m(1-\varepsilon)\right)+\frac{\beta(1-\varepsilon)S_h}{N_h}\left(\alpha(1-\omega)(\rho Z+vl)+kwXZ\right)+\frac{\beta\eta_2 S_h}{N_h}l\left(\alpha(1-\omega)Y+k\omega m(1-\varepsilon)\right)}{W\left(Z(\varpi\psi\varepsilon+\mu+\delta)X+m(1-\varepsilon)(\mu+\delta)(l+Z)\right)}$$

$$=\frac{\beta S_h}{N_h}\left(\frac{\eta_1 Z\left(\alpha(1-\omega)Y+k\omega m(1-\varepsilon)\right)+(1-\varepsilon)\left(\alpha(1-\omega)(\rho Z+vl)+kwXZ\right)+\eta_2 l\left(\alpha(1-\omega)Y+k\omega m(1-\varepsilon)\right)}{W\left(Z(\varpi\psi\varepsilon+\mu+\delta)X+m(1-\varepsilon)(\mu+\delta)(l+Z)\right)}\right)$$

for $0<\varepsilon,\varpi<1$

Thus,

$$R_e=\frac{\beta\Lambda}{\mu N_h}\left(\frac{\eta_1 Z\left(\alpha(1-\omega)Y+k\omega m(1-\varepsilon)\right)+(1-\varepsilon)\left(\alpha(1-\omega)(\rho Z+vl)+kwXZ\right)+\eta_2 l\left(\alpha(1-\omega)Y+k\omega m(1-\varepsilon)\right)}{W\left(Z(\varpi\psi\varepsilon+\mu+\delta)X+m(1-\varepsilon)(\mu+\delta)(l+Z)\right)}\right) \quad (6)$$

3.4 Local Stability of DFE

Theorem 2: The Disease Free Equilibrium point is said to be locally asymptotically stable if the effective reproduction number is less than unity $(R_e<1)$ and unstable if $R_e>1$.

Proof:

We obtained the Jacobian matrix of the system (1) at the DFE (ε_0) ,

$$J(\varepsilon_0)=\begin{bmatrix}-\mu & 0 & -\frac{\beta\eta_1\Lambda}{N_h\mu} & -\frac{\beta(1-\varepsilon)\Lambda}{N_h\mu} & 0 & -\frac{\beta\eta_2\Lambda}{N_h\mu}\\ 0 & -(k\varpi+a(1-\varpi)+\mu) & \frac{\beta\eta_1\Lambda}{N_h\mu} & \frac{\beta(1-\varepsilon)\Lambda}{N_h\mu} & 0 & \frac{\beta\eta_2\Lambda}{N_h\mu}\\ 0 & a(1-\varpi) & -(\rho+l+\mu+\delta) & m(1-\varepsilon) & 0 & 0\\ 0 & k\varpi & \rho & -(\varpi\psi\varepsilon+m(1-\varepsilon)+\mu+\delta) & 0 & v\\ 0 & 0 & 0 & \varpi\psi\varepsilon & -(\mu) & 0\\ 0 & 0 & l & 0 & 0 & -(v+\mu+\delta)\end{bmatrix}$$

Letting
$W=\left(k\varpi+a\left(1-\varpi\right)+\mu\right),\ X=\left(\rho+l+\mu+\delta\right),\ Y=\left(\varpi\psi\varepsilon+m\left(1-\varepsilon\right)+\mu+\delta\right),\ \ and\ Z=\left(v+\mu+\delta\right)$, the eigenvalues of the Jacobian matrix are given as;

$$\left\|\begin{bmatrix} -\mu-\lambda & 0 & -\frac{\beta\eta_1\Lambda}{N_h\mu} & -\frac{\beta(1-\varepsilon)\Lambda}{N_h\mu} & 0 & -\frac{\beta\eta_2\Lambda}{N_h\mu} \\ 0 & -W-\lambda & \frac{\beta\eta_1\Lambda}{N_h\mu} & \frac{\beta(1-\varepsilon)\Lambda}{N_h\mu} & 0 & \frac{\beta\eta_2\Lambda}{N_h\mu} \\ 0 & a(1-\varpi) & -X-\lambda & m(1-\varepsilon) & 0 & 0 \\ 0 & k\varpi & \rho & -Y-\lambda & 0 & v \\ 0 & 0 & 0 & \varpi\psi\varepsilon & -(\mu)-\lambda & 0 \\ 0 & 0 & l & 0 & 0 & -Z-\lambda \end{bmatrix}\right\|=0$$

$$\Rightarrow(-\mu-\lambda)(-\mu-\lambda)\left\|\begin{bmatrix} -W-\lambda & \frac{\beta\eta_1\Lambda}{N_h\mu} & \frac{\beta(1-\varepsilon)\Lambda}{N_h\mu} & \frac{\beta\eta_2\Lambda}{N_h\mu} \\ a(1-\varpi) & -X-\lambda & m(1-\varepsilon) & 0 \\ k\varpi & \rho & -Y-\lambda & v \\ 0 & l & 0 & -Z-\lambda \end{bmatrix}\right\|=0$$

This implies that

$$(-\mu-\lambda)(-\mu-\lambda)=0 \quad \text{or} \quad \left\|\begin{bmatrix} -W-\lambda & \frac{\beta\eta_1\Lambda}{N_h\mu} & \frac{\beta(1-\varepsilon)\Lambda}{N_h\mu} & \frac{\beta\eta_2\Lambda}{N_h\mu} \\ a(1-\varpi) & -X-\lambda & m(1-\varepsilon) & 0 \\ k\varpi & \rho & -Y-\lambda & v \\ 0 & l & 0 & -Z-\lambda \end{bmatrix}\right\|=0$$

But $(-\mu-\lambda)(-\mu-\lambda)=\lambda^2+2\lambda(\mu)+(\mu^2)\neq 0$
Since μ is a positive number.
Hence,

$$\begin{bmatrix} -W-\lambda & \frac{\beta\eta_1\Lambda}{N_h\mu} & \frac{\beta(1-\varepsilon)\Lambda}{N_h\mu} & \frac{\beta\eta_2\Lambda}{N_h\mu} \\ a(1-\varpi) & -X-\lambda & m(1-\varepsilon) & 0 \\ k\varpi & \rho & -Y-\lambda & v \\ 0 & l & 0 & -Z-\lambda \end{bmatrix}=0$$

$$\Rightarrow -(W+\lambda)\begin{vmatrix} -X-\lambda & m(1-\varepsilon) & 0 \\ \rho & -Y-\lambda & v \\ l & 0 & -Z-\lambda \end{vmatrix}-a(1-\varpi)\begin{vmatrix} \frac{\beta\eta_1\Lambda}{N_h\mu} & \frac{\beta(1-\varepsilon)\Lambda}{N_h\mu} & \frac{\beta\eta_2\Lambda}{N_h\mu} \\ \rho & -Y-\lambda & v \\ l & 0 & -Z-\lambda \end{vmatrix}+k\varpi\begin{vmatrix} \frac{\beta\eta_1\Lambda}{N_h\mu} & \frac{\beta(1-\varepsilon)\Lambda}{N_h\mu} & \frac{\beta\eta_2\Lambda}{N_h\mu} \\ -X-\lambda & m(1-\varepsilon) & 0 \\ l & 0 & -Z-\lambda \end{vmatrix}=0$$

after which the expansion and simplifications give the following polynomial;

$$\lambda^4+\lambda^3(W+X+Y+Z)+\lambda^2\left(WZ+WX-\frac{kw\beta(1-\varepsilon)\Lambda}{N_h\mu}+XY+XZ+YZ-\rho m(1-\varepsilon)+WY-\frac{a(1-\varpi)\beta\eta_1\Lambda}{N_h\mu}\right)$$
$$+\lambda\left(\begin{array}{l}\frac{a(1-\varpi)\rho\beta(1-\varepsilon)\Lambda}{N_h\mu}-\frac{a(1-\varpi)l\beta\eta_2\Lambda}{N_h\mu}-\frac{kw\beta\eta_1\Lambda m(1-\varepsilon)}{N_h\mu}-\frac{a(1-\varpi)\beta\eta_1\Lambda Y}{N_h\mu}-\frac{a(1-\varpi)\beta\eta_1\Lambda Z}{N_h\mu}\\ +WXY+WXZ+WYZ-\frac{k\varpi\beta(1-\varepsilon)\Lambda X}{N_h\mu}+XYZ-\rho m(1-\varepsilon)Z-lm(1-\varepsilon)v-W\rho m(1-\varepsilon)-\frac{k\varpi\beta(1-\varepsilon)\Lambda Z}{N_h\mu}\end{array}\right)$$
$$+WXYZ-W\rho m(1-\varepsilon)Z-Wlm(1-\varepsilon)v-\frac{a(1-\varpi)\beta\eta_1\Lambda YZ}{N_h\mu}-\frac{a(1-\varpi)\rho\beta(1-\varepsilon)\Lambda Z}{N_h\mu}-\frac{a(1-\varpi)l\beta(1-\varepsilon)\Lambda v}{N_h\mu}$$
$$-\frac{a(1-\varpi)l\beta\eta_2\Lambda Y}{N_h\mu}-\frac{k\varpi\beta\eta_1\Lambda m(1-\varepsilon)Z}{N_h\mu}-\frac{k\varpi X\beta(1-\varepsilon)\Lambda Z}{N_h\mu}-\frac{k\varpi l\beta\eta_2\Lambda m(1-\varepsilon)}{N_h\mu}$$
$$\Rightarrow\lambda^4+\lambda^3(W+X+Y+Z)+\lambda^2\left(WZ+WX+XY+XZ+YZ+WY-\frac{\beta\Lambda}{N_h\mu}\left(kw(1-\varepsilon)+a(1-\varpi)\eta_1\right)-\rho m(1-\varepsilon)\right)$$
$$+\lambda\left(\begin{array}{l}WXY+WXZ+WYZ+XYZ+\frac{\beta\Lambda}{N_h\mu}\left(a(1-\varpi)\left(\rho(1-\varepsilon)-l\eta_2-\eta_1Y-\eta_1Z\right)-k\varpi(1-\varepsilon)\left(\eta_1m+X+Z\right)\right)\\ -m(1-\varepsilon)\left(\rho Z-lv-W\rho\right)\end{array}\right)$$
$$+WXYZ-W\rho m(1-\varepsilon)Z-Wlm(1-\varepsilon)v-\frac{\beta\Lambda}{N_h\mu}\left(\begin{array}{l}a(1-\varpi)\left(\eta_1YZ+\rho(1-\varepsilon)Z+l(1-\varepsilon)v+l\eta_2Y\right)\\ +(1-\varepsilon)\left(mk\varpi\eta_1Z+k\varpi X(1-\varepsilon)Z+k\varpi ml\eta_2\right)\end{array}\right)=0 \qquad (7)$$

This can be expressed as $\lambda^4A_0+\lambda^3A_1+\lambda^2A_2+\lambda A_3+A_4=0$, where;

$$A_0=1,\quad A_1=W+X+Y+Z$$
$$A_2=WZ+WX+XY+XZ+YZ+WY-\frac{\beta\Lambda}{N_h\mu}\left(kw(1-\varepsilon)+a(1-\varpi)\eta_1\right)-\rho m(1-\varepsilon),$$
$$A_3=WXY+WXZ+WYZ+XYZ+\frac{\beta\Lambda}{N_h\mu}\left(a(1-\varpi)\left(\rho(1-\varepsilon)-l\eta_2-\eta_1Y-\eta_1Z\right)-k\varpi(1-\varepsilon)\left(\eta_1m+X+Z\right)\right)$$
$$-m(1-\varepsilon)\left(\rho Z-lv-W\rho\right),$$
$$A_4=WXYZ-W\rho m(1-\varepsilon)Z-Wlm(1-\varepsilon)v-\frac{\beta\Lambda}{N_h\mu}\left(\begin{array}{l}a(1-\varpi)\left(\eta_1YZ+\rho(1-\varepsilon)Z+l(1-\varepsilon)v+l\eta_2Y\right)\\ +(1-\varepsilon)\left(mk\varpi\eta_1Z+k\varpi X(1-\varepsilon)Z+mk\varpi l\eta_2\right)\end{array}\right)$$

By the Routh-Hurwitz criteria, the fourth-order polynomial equation as presented in (7) has strictly negative real part roots, if

$$\Delta_0=A_0>0,\Delta_1=A_1>0,\ \Delta_2=\begin{vmatrix}A_1 & A_0\\ A_3 & A_2\end{vmatrix}=A_1A_2-A_3A_0>0$$

$$\Delta_3=\begin{vmatrix}A_1 & A_0 & 0\\ A_3 & A_2 & A_1\\ 0 & A_4 & A_3\end{vmatrix}=A_1A_2A_3-A_1{}^2A_4-A_0A_3{}^2>0,\Delta_4>0$$

Otherwise ε_0 is unstable.

It is already established that $\Delta_0 = 1, \Delta_1 > 0$ since all the parameters are positive.

$$\Delta_2 = A_1 A_2 - A_3 A_0 > 0$$

$$\Rightarrow (W+X+Y+Z)\left(WZ+WX+XY+XZ+YZ+WY-\frac{\beta\Lambda}{N_h\mu}\left(kw(1-\varepsilon)+a(1-\varpi)\eta_1\right)-\rho m(1-\varepsilon)\right)$$

$$-\left(\begin{array}{l} WXY+WXZ+WYZ+XYZ+\frac{\beta\Lambda}{N_h\mu}\left(a(1-\varpi)(\rho(1-\varepsilon)-l\eta_2-\eta_1 Y-\eta_1 Z)-k\varpi(1-\varepsilon)(\eta_1 m+X+Z)\right) \\ \quad -m(1-\varepsilon)(\rho Z-lv-W\rho), \end{array}\right)>0$$

$$\Rightarrow (W+X+Y+Z)\left(WZ+WX+XY+XZ+YZ+WY-\frac{\beta\Lambda}{N_h\mu}\left(kw(1-\varepsilon)+a(1-\varpi)\eta_1\right)-\rho m(1-\varepsilon)\right)$$

$$>\left(\begin{array}{l} WXY+WXZ+WYZ+XYZ+\frac{\beta\Lambda}{N_h\mu}\left(a(1-\varpi)(\rho(1-\varepsilon)-l\eta_2-\eta_1 Y-\eta_1 Z)-k\varpi(1-\varepsilon)(\eta_1 m+X+Z)\right) \\ \quad -m(1-\varepsilon)(\rho Z-lv-W\rho), \end{array}\right)$$

$$\Delta_3 = A_1 A_2 A_3 - A_1^{\,2} A_4 - A_0 A_3^{\,2} > 0,$$

$$\Rightarrow \Delta_3 = A_1 A_2 A_3 > A_1^{\,2} A_4 + A_0 A_3^{\,2}$$

Also,

$$\Delta_4 > 0$$

$$\Rightarrow WXYZ - W\rho m(1-\varepsilon)Z - Wlm(1-\varepsilon)v - \frac{a(1-\varpi)\beta\eta_1\Lambda YZ}{N_h\mu} - \frac{a(1-\varpi)\rho\beta(1-\varepsilon)\Lambda Z}{N_h\mu} - \frac{a(1-\varpi)l\beta(1-\varepsilon)\Lambda v}{N_h\mu}$$

$$-\frac{a(1-\varpi)l\beta\eta_2\Lambda Y}{N_h\mu} - \frac{k\varpi\beta\eta_1\Lambda m(1-\varepsilon)Z}{N_h\mu} - \frac{k\varpi X\beta(1-\varepsilon)\Lambda m(1-\varepsilon)Z}{N_h\mu} - \frac{k\varpi l\beta\eta_2\Lambda m(1-\varepsilon)}{N_h\mu} > 0$$

$$\Rightarrow W\left(XYZ-\rho m(1-\varepsilon)Z-lm(1-\varepsilon)v\right)-\frac{\beta\Lambda}{N_h\mu}\left(\begin{array}{l} a(1-\varpi)\eta_1 YZ+a(1-\varpi)\rho(1-\varepsilon)Z+a(1-\varpi)l(1-\varepsilon)v+a(1-\varpi)l\eta_2 Y \\ +k\varpi\eta_1 m(1-\varepsilon)Z+k\varpi X(1-\varepsilon)Z+k\varpi l\eta_2 m(1-\varepsilon) \end{array}\right)>0$$

$$\Rightarrow W\left(XYZ-\rho m(1-\varepsilon)Z-lm(1-\varepsilon)v\right)$$

$$>\frac{\beta\Lambda}{N_h\mu}\left(\eta_1 Z\left(a(1-\varpi)Y+k\varpi m(1-\varepsilon)\right)+(1-\varepsilon)\left(\left(a(1-\varpi)(\rho Z+lv)+k\varpi XZ\right)\right)+l\eta_2\left(a(1-\varpi)Y+k\varpi m(1-\varepsilon)\right)\right)$$

$$\Rightarrow \frac{\beta\Lambda}{\mu N_h}\left(\frac{\eta_1 Z\left(\alpha(1-\omega)Y+k\omega m(1-\varepsilon)\right)+(1-\varepsilon)\left(\alpha(1-\omega)(\rho Z+vl)+kwXZ\right)+\eta_2 l\left(\alpha(1-\omega)Y+k\omega m(1-\varepsilon)\right)}{W\left(XYZ-\rho m(1-\varepsilon)Z-lvm(1-\varepsilon)\right)}\right)<1$$

$$\Rightarrow R_e < 1$$

Thus, the DFE of system (1) is locally asymptotically stable if $R_e < 1$. This shows that HIV can be eliminated from the community. This also implies that the initial population sizes in the model are in the basin of attraction of ε_0.

3.5 Endemic equilibrium

The endemic equilibrium state is calculated by equating each of the equations of (1) to zero and solving simultaneously for each of the state variables. This is a state where HIV/AIDS is endemic in the population, which we shall denote as $\varepsilon_h = \left(S_h^*, E_h^*, I_h^*, T_h^*, U_h^*, A_h^*\right)$

$$\frac{dS_h^*}{dt} = \Lambda - \left(\xi^* + \mu\right)S_h^* = 0$$
$$\frac{dE_h^*}{dt} = \xi^* S_h^* - \left(k\varpi + \alpha(1-\varpi) + \mu\right)E_h^* = 0$$
$$\frac{dI_h^*}{dt} = \alpha(1-\varpi)E_h^* + m(1-\varepsilon)T_h^* - (\rho + l + \mu + \delta)I_h^* = 0$$
$$\frac{dT_h^*}{dt} = k\varpi E_h^* + \rho I_h^* + \nu A_h^* - \left(\varpi\psi\varepsilon + m(1-\varepsilon) + \mu + \delta\right)T_h^* = 0$$
$$\frac{dU_h^*}{dt} = \varpi\psi\varepsilon T^*{}_h - \mu U_h^* = 0 \qquad (8)$$
$$\frac{dA_h^*}{dt} = lI_h^* - (\nu + \mu + \delta)A_h^* = 0$$

The endemic equilibrium point is given as;

$$S_h^* = \frac{\Lambda}{\left(\xi^* + \mu\right)}, \qquad E_h^* = \frac{\xi^* S_h^*}{W}, \qquad I_h^* = \frac{\alpha(1-\varpi)E_h^* + m(1-\varepsilon)T_h^*}{X}$$

$$T_h^* = \frac{k\varpi E_h^* + \rho I_h^* + \nu A_h^*}{Y}, \qquad U_h^* = \frac{\varpi\psi\varepsilon T_h^*}{\mu}, \qquad A_h^* = \frac{lI_h^*}{Z} \qquad (9)$$

Further substitutions and working gives;

$$T_h^* = \frac{k\varpi\dfrac{\xi^* S_h^*}{W} + \rho I_h^* + \nu A_h^*}{Y}$$

$$\Rightarrow T_h^* = \frac{XZk\varpi\xi^* S_h^* + \left(\rho Z\alpha(1-\varpi)\xi^* S_h^* + \rho ZWm(1-\varepsilon)T_h^*\right) + \left(\nu l\alpha(1-\varpi)\xi^* S_h^* + \nu lWm(1-\varepsilon)T_h^*\right)}{WXYZ}$$

$$T_h^*\left((WXZY) - Wm(1-\varepsilon)(\rho Z + \nu l)\right) = \xi^* S_h^*\left(XZk\varpi + \rho Z\alpha(1-\varpi) + \nu l\alpha(1-\varpi)\right)$$

$$T_h^* = \frac{\xi^*\Lambda\left(XZk\varpi + \rho Z\alpha(1-\varpi) + \nu l\alpha(1-\varpi)\right)}{W\left(\xi^* + \mu\right)\left(XYZ - m(1-\varepsilon)(\rho Z + \nu l)\right)}$$

$$I_h^* = \frac{\xi^* \Lambda\left(\alpha(1-\varpi)\left((XZY) - m(1-\varepsilon)(\rho Z + \nu l)\right) + m(1-\varepsilon)(XZk\varpi + \rho Z\alpha(1-\varpi) + \nu l\alpha(1-\varpi))\right)}{WX\left(\xi^* + \mu\right)(XYZ - m(1-\varepsilon)(\rho Z + \nu l))}$$

$$A_h^* = \frac{l\xi^* \Lambda\left(\alpha(1-\varpi)\left((XZY) - m(1-\varepsilon)(\rho Z + \nu l)\right) + m(1-\varepsilon)(XZk\varpi + \rho Z\alpha(1-\varpi) + \nu l\alpha(1-\varpi))\right)}{WXZ\left(\xi^* + \mu\right)(XYZ - m(1-\varepsilon)(\rho Z + \nu l))}$$

Substituting in the force of infection gives;

$$\xi^* = \frac{\beta\left((1-\varepsilon)T_h^* + \eta_1 I_h^* + \eta_2 A_h^*\right)}{N_h}$$

$$= \frac{\beta\left(\begin{array}{l}(1-\varepsilon)\dfrac{\xi^* \Lambda\left(XZk\varpi + \rho Z\alpha(1-\varpi) + \nu l\alpha(1-\varpi)\right)}{W\left(\xi^* + \mu\right)(XYZ - m(1-\varepsilon)(\rho Z + \nu l))} \\ +\eta_1 \dfrac{\xi^* \Lambda\left(\alpha(1-\varpi)\left((XZY) - m(1-\varepsilon)(\rho Z + \nu l)\right) + m(1-\varepsilon)(XZk\varpi + \rho Z\alpha(1-\varpi) + \nu l\alpha(1-\varpi))\right)}{WX\left(\xi^* + \mu\right)(XYZ - m(1-\varepsilon)(\rho Z + \nu l))} \\ +\eta_2 \dfrac{l\xi^* \Lambda\left(\alpha(1-\varpi)\left((XZY) - m(1-\varepsilon)(\rho Z + \nu l)\right) + m(1-\varepsilon)(XZk\varpi + \rho Z\alpha(1-\varpi) + \nu l\alpha(1-\varpi))\right)}{WXZ\left(\xi^* + \mu\right)(XYZ - m(1-\varepsilon)(\rho Z + \nu l))}\end{array}\right)}{N_h}$$

$$\xi^* = \frac{\beta\left(\begin{array}{l}(1-\varepsilon)\dfrac{\xi^* \Lambda\left(XZk\varpi + \rho Z\alpha(1-\varpi) + \nu l\alpha(1-\varpi)\right)}{W\left(\xi^* + \mu\right)(XYZ - m(1-\varepsilon)(\rho Z + \nu l))} \\ +\eta_1 \dfrac{\xi^* \Lambda\left(\alpha(1-\varpi)\left((XZY) - m(1-\varepsilon)(\rho Z + \nu l)\right) + m(1-\varepsilon)(XZk\varpi + \rho Z\alpha(1-\varpi) + \nu l\alpha(1-\varpi))\right)}{WX\left(\xi^* + \mu\right)(XYZ - m(1-\varepsilon)(\rho Z + \nu l))} \\ +\eta_2 \dfrac{l\xi^* \Lambda\left(\alpha(1-\varpi)\left((XZY) - m(1-\varepsilon)(\rho Z + \nu l)\right) + m(1-\varepsilon)(XZk\varpi + \rho Z\alpha(1-\varpi) + \nu l\alpha(1-\varpi))\right)}{WXZ\left(\xi^* + \mu\right)(XYZ - m(1-\varepsilon)(\rho Z + \nu l))}\end{array}\right)}{N_h}$$

$$\left(\xi^*\right)=\frac{\beta\Lambda\left(\begin{array}{l}(1-\varepsilon)XZ\left(XZk\varpi+\rho Z\alpha(1-\varpi)+\nu l\alpha(1-\varpi)\right)\\+\eta_1\left(\alpha(1-\varpi)Z\left((XZY)-m(1-\varepsilon)(\rho Z+\nu l)\right)+m(1-\varepsilon)\left(XZk\varpi+\rho Z\alpha(1-\varpi)+\nu l\alpha(1-\varpi)\right)\right)\\+\eta_2 l\left(\alpha(1-\varpi)\left((XZY)-m(1-\varepsilon)(\rho Z+\nu l)\right)+m(1-\varepsilon)\left(XZk\varpi+\rho Z\alpha(1-\varpi)+\nu l\alpha(1-\varpi)\right)\right)\end{array}\right)}{N_h WXZ\left(XYZ-m(1-\varepsilon)(\rho Z+\nu l)\right)}-\frac{\mu WXZ\left(XYZ-m(1-\varepsilon)(\rho Z+\nu l)\right)}{WXZ\left(XYZ-m(1-\varepsilon)(\rho Z+\nu l)\right)}$$

$$\xi^*=\frac{\beta\Lambda\left(\begin{array}{l}\left(X^2Z^2(1-\varepsilon)k\varpi+XZ(1-\varepsilon)\alpha(1-\varpi)(\rho Z+\nu l)\right)\\+\eta_1 Z\left(\alpha(1-\varpi)\left((XZY)+XZm(1-\varepsilon)k\varpi\right)\right)\\+\eta_2 l\left(\alpha(1-\varpi)(XZY)+XZm(1-\varepsilon)k\varpi\right)\end{array}\right)}{N_h WXZ\left(XYZ-m(1-\varepsilon)(\rho Z+\nu l)\right)}-\mu$$

$$\xi^*=\mu\left(\frac{\beta\Lambda\left(\left(XZ(1-\varepsilon)k\varpi+(1-\varepsilon)\alpha(1-\varpi)(\rho Z+\nu l)\right)+\eta_1 Z\left(\alpha(1-\varpi)\left((Y)+m(1-\varepsilon)k\varpi\right)\right)+\eta_2 l\left(\alpha(1-\varpi)(Y)+m(1-\varepsilon)k\varpi\right)\right)}{\mu N_h W\left(XYZ-m(1-\varepsilon)(\rho Z+\nu l)\right)}-1\right)$$

$$\xi^*=\mu\left(R_\varepsilon-1\right) \tag{10}$$

Thus, the force of infection at steady-state ξ^* is positive only if $R_\varepsilon>1$.

Therefore, the model system (1) has a unique endemic equilibrium whenever $R_\varepsilon>1$

4. Optimal control model

In our optimal control of HIV infection, we extend our model (1) by incorporating three distinct time-dependent control measures. Optimal control analysis helps identify the best control measure to adopt in the eradication or control of the disease in the community at a specified period (Cheneke K.R., 2023; Ayele, T.K., 2021; Cheneke K.R. *et al,* 2021; Marsudi, 2018). The three single control strategies considered are;

1. Pre-contact preventive measures u_1, such as those that protect the susceptible from contracting HIV infection.
2. Post-contact preventive measure u_2, aimed at protecting from contracting HIV infection after an effective exposure to the disease. This is the use of Post-Exposure Prophylaxis (PEP) intervention. This is usually commenced within 72 hours after an effective exposure to the virus to prevent infection (CDC 2015).
3. Treatment control measures u_3, are used on pre-AIDS and AIDS individuals to reduce the viral load and stop or slow the progression.

Incorporating these control measures in our model (1) gives:

$$\begin{cases} \dfrac{dS_h}{dt} = \Lambda - \left(\left(1-u_1(t)\right) \dfrac{\beta\left(\left(1-\varepsilon\right)T_h + \eta_1 I_h + \eta_2 A_h\right)}{N_h} + \mu \right) S_h \\ \dfrac{dE_h}{dt} = \left(1-u_1(t)\right) \dfrac{\beta\left(\left(1-\varepsilon\right)T_h + \eta_1 I_h + \eta_2 A_h\right)}{N_h} S_h - \left(k u_2(t) + \alpha\left(1-u_2(t)\right) + \mu\right) E_h \\ \dfrac{dI_h}{dt} = \alpha\left(1-u_2(t)\right) E_h + m\left(1-\varepsilon\right) T_h - \left(u_3(t)\rho + \left(1-u_3(t)\right) l + \mu + \delta\right) I_h \\ \dfrac{dT_h}{dt} = k u_2(t) E_h + u_3(t)\rho I_h + u_3(t)\nu A_h - \left(u_2(t)\psi\varepsilon + m\left(1-\varepsilon\right) + \mu + \delta\right) T_h \\ \dfrac{dU_h}{dt} = u_2(t)\psi\varepsilon T_h - \left(\mu\right) U_h \\ \dfrac{dA_h}{dt} = \left(1-u_3(t)\right) l I_h - \left(u_3(t)\nu + \mu + \delta\right) A_h \end{cases} \tag{11}$$

where $\varpi = u_2$ and with the initial conditions, $S_h > 0, E_h > 0, I_h > 0, T_h > 0, U_h > 0, A_h > 0$
AIDS patients are treated with ART to have their HIV viral loads lowered and progress to the Treatment class at the rate ν

Our goal is to minimize the population of the under-3-day Exposed (E_h), the pre-AIDS population (I_h), the population on treatment (T_h), and the AIDS (A_h) population. We also, seek to increase the number of those who were Uninfected (U_h) , after an exposure to the virus, and consider the cost involved in using these control strategies.

From Pontryagin (1918) and Fleming (2012), we have the objective function defined as:

$$J(u_1,u_2,u_3) = \int_0^{tf} \left(B_1 E_h + B_2 I_h + B_3 T_h + B_4 A_h + C_1 \frac{u_1^{\,2}}{2} + C_2 \frac{u_2^{\,2}}{2} + C_3 \frac{u_3^{\,2}}{2} \right) dt, \tag{12}$$

where B_1, B_2, B_3, B_4 are all positive weight constants of the under 3 days-Exposed, pre-AIDS, Treatment and AIDS classes, respectively. Also, $C_1 \frac{u_1^{\,2}}{2}, C_2 \frac{u_2^{\,2}}{2}$ and $C_3 \frac{u_3^{\,2}}{2}$ are the cost control functions. The constants C_1, C_2, C_3 are measures of the relative cost of the interventions associated with the controls u_i (Seidu Baba and Makinde Oluwole, 2014). The cost function is quadratic because it is nonlinear (Silva, C.J., Torres, D.F., 2017). We aim to find the optimal control of $u_1^{\,*}(t), u_2^{\,*}(t)$ and $u_3^{\,*}(t)$ such that

$J\left(u_1^{\,*}(t), u_2^{\,*}(t), u_3^{\,*}(t)\right) = \min\{(u_1,u_2,u_3) : u_1,u_2,u_3 \in \Delta\}$, where Δ is a non-empty control, subject to the optimal control model set defined as $\Delta = \{u_i : 0 \le u_i(t) \le 1$, Lebesgue

measurable $t=[0,t_f]$ for $i=1,2,3\}$, representing the control set with respect to the initial conditions.

To show that the optimal control problem exists, we employ Pontryagin's maximum principle (Pontryagin L.S, 1987), which provides the necessary condition for the existence optimal controls $u^*=(u_1,u_2,u_3)$. By this principle, we have the Hamiltonian function (H) defined as;

$$H(S_h,E_h,I_h,T_h,U_h,A_h)=B_1E_h+B_2I_h+B_3T_h+B_4A_h+\frac{1}{2}\left(C_1{u_1}^2+C_2{u_2}^2+C_3{u_3}^2\right)+\lambda_1\frac{dS_h}{dt}+\lambda_2\frac{dE_h}{dt}+\lambda_3\frac{dI_h}{dt}+\lambda_4\frac{dT_h}{dt}+\lambda_5\frac{dU_h}{dt}+\lambda_6\frac{dA_h}{dt} \quad (13)$$

Where $\lambda_i, i=1,2,3,4,5,6$ are the adjoint variables associated with the state variables of the optimal control model (11)

According to Peter O. *et al.* (2020), we establish the following results.

Theorem 3 Let ${S_h}^*,{E_h}^*,{I_h}^*,{T_h}^*,{U_h}^*,{A_h}^*$, be the optimal state solutions associated with the optimal control ${u_1}^*,{u_2}^*,{u_3}^*$, for the optimal problem in (11). There exist adjoint functions λ_i , which satisfy the system (13) with the transversal conditions $\lambda_i(t_f)=0$, for $i=1,2,3,3,4,5,6$, and the control variables $\left({u_1}^*,{u_2}^*,{u_3}^*\right)$.

Proof:

From the second condition of Pontryagin's maximum principle (PMP), there exist adjoint variables, λ_i , $i=1,2,3,3,4,5,6$ which satisfy the following canonical equations: $\frac{d\lambda_1}{dt}=-\frac{dH}{dS_h}$,

$\frac{d\lambda_2}{dt}=-\frac{dH}{dE_h}$, $\frac{d\lambda_3}{dt}=-\frac{dH}{dI_h}$, $\frac{d\lambda_4}{dt}=-\frac{dH}{dT_h}$, $\frac{d\lambda_5}{dt}=-\frac{dH}{dU_h}$, and $\frac{d\lambda_6}{dt}=-\frac{dH}{dA_h}$

Differentiating the Hamiltonian function (13) with respect to the state variables S_h,E_h,I_h,T_h,U_h and A_h, we have:

$$\frac{d\lambda_1}{dt}=-\frac{\partial H}{\partial S}=-\left(\begin{array}{l}-\lambda_1(1-u_1(t))\left(\frac{\beta((1-\varepsilon)T_h+\eta_1I_h+\eta_2A_h)}{N_h}-\frac{\beta((1-\varepsilon)T_h+\eta_1I_h+\eta_2A_h)S}{{N_h}^2}\right)-\lambda_1\mu\\+\lambda_2\left((1-u_1(t))\frac{\beta((1-\varepsilon)T_h+\eta_1I_h+\eta_2A_h)}{N_h}-\frac{\beta((1-\varepsilon)T_h+\eta_1I_h+\eta_2A_h)S}{{N_h}^2}\right)\end{array}\right)$$

$$=(\lambda_1-\lambda_2)(1-u_1(t))\left(\frac{\beta((1-\varepsilon)T_h+\eta_1I_h+\eta_2A_h)}{N_h}-\frac{\beta((1-\varepsilon)T_h+\eta_1I_h+\eta_2A_h)S}{{N_h}^2}\right)+\lambda_1\mu$$

$$=(\lambda_1-\lambda_2)(1-u_1(t))\left(\frac{\beta((1-\varepsilon)T_h+\eta_1I_h+\eta_2A_h)}{N_h}\right)\left(1-\frac{S}{N_h}\right)+\lambda_1\mu$$

$$\frac{d\lambda_2}{dt} = -\frac{\partial H}{\partial E} = -\begin{pmatrix} (B_1) + \lambda_1(1-u_1(t))\dfrac{\beta((1-\varepsilon)T_h + \eta_1 I_h + \eta_2 A_h)S}{N_h^2} - \lambda_2(1-u_1(t))\dfrac{\beta((1-\varepsilon)T_h + \eta_1 I_h + \eta_2 A_h)S}{N_h^2} \\ -\lambda_2(ku_2(t) + \alpha(1-u_2(t)) + \mu) + \lambda_3\alpha(1-u_2(t)) + \lambda_4 ku_2(t) \end{pmatrix}$$

$$= -B_1 + (\lambda_2 - \lambda_1)(1-u_1(t))\frac{\beta((1-\varepsilon)T_h + \eta_1 I_h + \eta_2 A_h)S}{N_h^2} + (\lambda_2 - \lambda_4)kuu_2(t) + (\lambda_2 - \lambda_3)\alpha(1-u_2(t)) + \lambda_2\mu$$

$$\frac{d\lambda_3}{dt} = -\frac{\partial H}{\partial I} = -\begin{pmatrix} (B_2) - \lambda_1(1-u_1(t))\left(\dfrac{\beta(\eta_1)S}{N_h} - \dfrac{\beta((1-\varepsilon)T_h + \eta_1 I_h + \eta_2 A_h)S}{N_h^2}\right) \\ + \lambda_2(1-u_1(t))\left(\dfrac{\beta(\eta_1)S}{N_h} - \dfrac{\beta((1-\varepsilon)T_h + \eta_1 I_h + \eta_2 A_h)S}{N_h^2}\right) - \lambda_3(u_3(t)\rho + (1-u_3(t))l + \mu + \delta) \\ + \lambda_4 u_3(t)\rho + \lambda_6(1-u_3(t))l \end{pmatrix}$$

$$= -B_2 + (\lambda_1 - \lambda_2)(1-u_1(t))\frac{\beta S}{N_h}\left(\eta_1 - \frac{((1-\varepsilon)T_h + \eta_1 I_h + \eta_2 A_h)}{N_h}\right)$$

$$+ (\lambda_3 - \lambda_4)u_3(t)\rho + (\lambda_3 - \lambda_6)(1-u_3(t))l + \lambda_3(\mu + \delta)$$

$$\frac{d\lambda_4}{dt} = -\frac{\partial H}{\partial T_h} = -\begin{pmatrix} B_3 - \lambda_1(1-u_1(t))\left(\dfrac{\beta(1-\varepsilon)S}{N_h} - \dfrac{\beta((1-\varepsilon)T_h + \eta_1 I_h + \eta_2 A_h)S}{N_h^2}\right) \\ + \lambda_2(1-u_1(t))\left(\dfrac{\beta(1-\varepsilon)S}{N_h} - \dfrac{\beta((1-\varepsilon)T_h + \eta_1 I_h + \eta_2 A_h)S}{N_h^2}\right) \\ + \lambda_3 m(1-\varepsilon) - \lambda_4(u_2(t)\psi\varepsilon + m(1-\varepsilon) + \mu + \delta) + \lambda_5 u_2(t)\psi\varepsilon \end{pmatrix}$$

$$= -B_3 + (\lambda_1 - \lambda_2)(1-u_1(t))\frac{\beta S}{N_h}\left((1-\varepsilon) - \frac{((1-\varepsilon)T_h + \eta_1 I_h + \eta_2 A_h)}{N_h}\right)$$

$$+ (\lambda_4 - \lambda_3)m(1-\varepsilon) + (\lambda_4 - \lambda_5)u_2(t)\psi\varepsilon + \lambda_4(\mu + \delta)$$

$$\frac{d\lambda_5}{dt} = -\frac{\partial H}{\partial U_h} = -\begin{pmatrix} -\lambda_1(1-u_1(t))\dfrac{\beta((1-\varepsilon)T_h + \eta_1 I_h + \eta_2 A_h)S}{N_h^2} + \\ \lambda_2(1-u_1(t))\dfrac{\beta((1-\varepsilon)T_h + \eta_1 I_h + \eta_2 A_h)S}{N_h^2} - \lambda_5\mu \end{pmatrix}$$

$$= (\lambda_1 - \lambda_2)(1-u_1(t))\frac{\beta((1-\varepsilon)T_h + \eta_1 I_h + \eta_2 A_h)S}{N_h^2} + \lambda_5\mu$$

$$\frac{d\lambda_6}{dt} = -\frac{\partial H}{\partial A_h} = -\left(\begin{array}{l} B_4 - \lambda_1(1-u_1(t))\left(\frac{\beta\eta_2 S}{N_h} - \frac{\beta((1-\varepsilon)T_h + \eta_1 I_h + \eta_2 A_h)S}{N_h{}^2}\right) \\ + \lambda_2(1-u_1(t))\left(\frac{\beta\eta_2 S}{N_h} - \frac{\beta((1-\varepsilon)T_h + \eta_1 I_h + \eta_2 A_h)S}{N_h{}^2}\right) + \lambda_4 u_3(t)\nu - \lambda_6(u_3(t)\nu + \mu + \delta) \end{array}\right)$$

$$= -B_4 + (\lambda_1 - \lambda_2)(1-u_1(t))\frac{\beta S}{N_h}\left(\eta_2 - \frac{((1-\varepsilon)T_h + \eta_1 I_h + \eta_2 A_h)}{N_h}\right) + (\lambda_6 - \lambda_4)u_3(t)\nu + \lambda_6(\mu + \delta)$$

with transversal conditions $\lambda_i(t_f) = 0,\ i = 1,2,3,3,4,5,6$

We establish the optimal control of the control variable set, where $u_i = (0,1),\quad i = 1,2,3$ by differentiating the Hamiltonian H in (13) at the steady state, with respect to the control variables (u_1, u_2, u_3). That is $\frac{dH}{du_1}|_{u_1=u_1{}^*} = 0$, $\frac{dH}{du_2}|_{u_2=u_2{}^*} = 0$, $\frac{dH}{du_3}|_{u_3=u_3{}^*} = 0$

$$\frac{dH}{du_1} = C_1 u_1 + \lambda_1 \frac{\beta((1-\varepsilon)T_h + \eta_1 I_h + \eta_2 A_h)S}{N_h} - \lambda_2 \frac{\beta((1-\varepsilon)T_h + \eta_1 I_h + \eta_2 A_h)S}{N_h} = 0$$

$$u_1{}^* = (\lambda_2 - \lambda_1)\frac{\beta((1-\varepsilon)T_h + \eta_1 I_h + \eta_2 A_h)S}{C_1 N_h}$$

$$\frac{dH}{du_2} = C_2 u_2 - \lambda_2(k-\alpha)E_h - \lambda_3 \alpha E_h + \lambda_4(kE_h - \psi\varepsilon T_h) + \lambda_5 \psi\varepsilon T_h = 0$$

$$\Rightarrow u_2{}^* = \frac{(\lambda_3 - \lambda_2)\alpha E_h + (\lambda_2 - \lambda_4)kE_h + (\lambda_4 - \lambda_5)\psi\varepsilon T_h}{C_2}$$

$$\frac{dH}{du_3} = C_3 u_3 - \lambda_3(\rho - l)I_h + \lambda_4(\rho I_h + \nu A_h) - \lambda_6(lI_h + \nu A_h) = 0$$

$$\Rightarrow u_3{}^* = \frac{(\lambda_3 - \lambda_4)\rho I_h + (\lambda_6 - \lambda_3)lI_h + (\lambda_6 - \lambda_4)\nu A_h}{C_3}$$

Therefore:

$$u_1{}^* = \max\left\{0, \left((\lambda_2 - \lambda_1)\frac{\beta((1-\varepsilon)T_h + \eta_1 I_h + \eta_2 A_h)S}{C_1 N_h}\right)\right\}$$

$$u_2{}^* = \max\left\{0, \left(\frac{(\lambda_3 - \lambda_2)\alpha E_h + (\lambda_2 - \lambda_4)kE_h + (\lambda_4 - \lambda_5)\psi\varepsilon T_h}{C_2}\right)\right\}$$

$$u_3{}^* = \max\left\{0, \left(\frac{(\lambda_3 - \lambda_4)\rho I_h + (\lambda_6 - \lambda_3)lI_h + (\lambda_6 - \lambda_4)\nu A_h}{C_3}\right)\right\}$$

5. Numerical simulation of the optimal control model and Discussion.

In this section, we investigate the impact of different optimal control measures and combined control measures to find the one that can quickly eradicate or curtail the spread of HIV/AIDS. The most effective cost strategy suitable for implementation among the various combinations

of these control strategies is discussed. This is very necessary because disease eradication in a given population is always cost-intensive. When recommending an intervention strategy for disease eradication, cost-effectiveness information on the interventions is needed for the optimal allocation of available resources.

Numerical simulations have been useful in determining the impact of these control measures on an ideal control model. It is useful in describing the dynamical behavior of a given system over time for different parameter values and conditions. Thus, we use Numerical simulation to illustrate the effect of these control measures and their combined strategies, using fourth-order Runge-Kutta forward–backward sweep simulations.

To this effect, we investigate the effect of optimal control strategies on the spread of HIV with the following categorization:

Strategy 1: Pre-contact prevention strategy (u_1)

Strategy 2: Post-contact prevention strategy (u_2)

Strategy 3: Control effort using only Anti-Retroviral treatment on the infected (u_3)

Paired strategies (4, 5, and 6): These are control efforts by pairing the strategies 1, 2, and 3. They are; (u_1,u_2), $(u_1,u_3)$and (u_2,u_3) respectively.

Strategy 7: Applying the first three strategies simultaneously (u_1,u_2,u_2)

The simulation's initial conditions were taken from Tigabu Kasia Ayele et al (2021), which are the approximated data of the HIV/AIDS rampage for the year 2019 in Ethiopia. The initial size of the populations has a fixed final time, $t_f = 10 years$. $S_h(0) = 1{,}000{,}000$, $E_h(0) = 144{,}900$, $I_h(0) = 363{,}400$, $T_h(0) = 448{,}000$, $U_h(0) = 500$(Assumed), $A_h(0) = 181{,}700$ representing a total population of $N = 2{,}138{,}500$.

Table 2 contains the parameter values used in the simulation, taken from the references cited therein. We have assumed some of them just for numerical purposes.

Table 2: Model parameter values.

Parameters	Values	Source
Λ	2000	Assumed
β	0.866/year	(Sharp P. M. and Hahn B. H. 2021)
ε	0.02	Assumed
η_1	0.8	(Kassa Semu Mitiku, Ouhinou Aziz 2021)
η_2	1.05	(Kassa Semu Mitiku, Ouhinou Aziz 2021)
μ	0.02/year	(Nyabadza F. et al 2010)

l	0.1/year	(Campos C. and Silva C.J.,2010)
m	0.75/year	(Maimunah and Dipo Aldila 2018)
δ	1/year	(Waziri A.S. *et al* 2012)
ψ	0.8	(NIH 2024)
α	0.333	(Silva C. J. and Torres D. 2017)
v	0.1	(Waziri A.S. *et al* 2012)
ρ	0.5/year	(Cheneke K.R. 2023)
k	0.03	Assumed

Unfortunately, we do not have good data on the coefficients of the costs associated with the control variables and infected persons. However, while we assume the positive wait constants in the objective function to be equal to ensure that none of the four terms is dominant during simulation, we cannot say the same for weight constants relating to the costs of control implementation. The cost associated with u_1, includes the cost of enlightenment through different channels like public campaigns, media, schools, and health centers. Cost of the provision of PrEP drugs and other protective devices like condoms. u_2 requires most of the cost associated with u_1 and cost of PEP, instead of PrEP drugs, in every health center and facility, which should be readily available and subsidized for the populace. u_3's cost is associated with the cost of enlightenment, ART drugs, and hospitalization. Thus, we assume $B_1 = B_2 = B_3 = B_4 = 1$ and $C_1 = 20, C_2 = 15, C_3 = 18$. Note that the weight constants are chosen based on assumptions and, as such, can fluctuate depending on different countries' realities.

Table 3 shows the possible classifications of these control strategies under the outlined control measures.

Table 3: Control Measures

Control Measures	Strategies
Single control:	Strategy 1: $u_1 \neq 0, u_2 = 0, u_3 = 0$ Strategy 2: $u_1 = 0, u_2 \neq 0, u_3 = 0$ Strategy 3: $u_1 = 0, u_2 = 0, u_3 \neq 0$
Double Control	Strategy 4: $u_1 \neq 0, u_2 \neq 0, u_3 = 0$ Strategy 5: $u_1 \neq 0, u_2 = 0, u_3 \neq 0$ Strategy 6: $u_1 = 0, u_2 \neq 0, u_3 \neq 0$

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Triple Control	Strategy 7: $u_1 \neq 0, u_2 \neq 0, u_3 \neq 0$

As shown in Table 3, each control measure is further divided into various strategies. We therefore simulate these strategies under each of the control measures. In our simulations, the under- 3-day HIV-Exposed class is referred to as the Exposed Class.

Strategy 1: The optimal use of pre-contact preventive strategies (u_1) only, for the control of transmission of HIV/AIDS in humans.

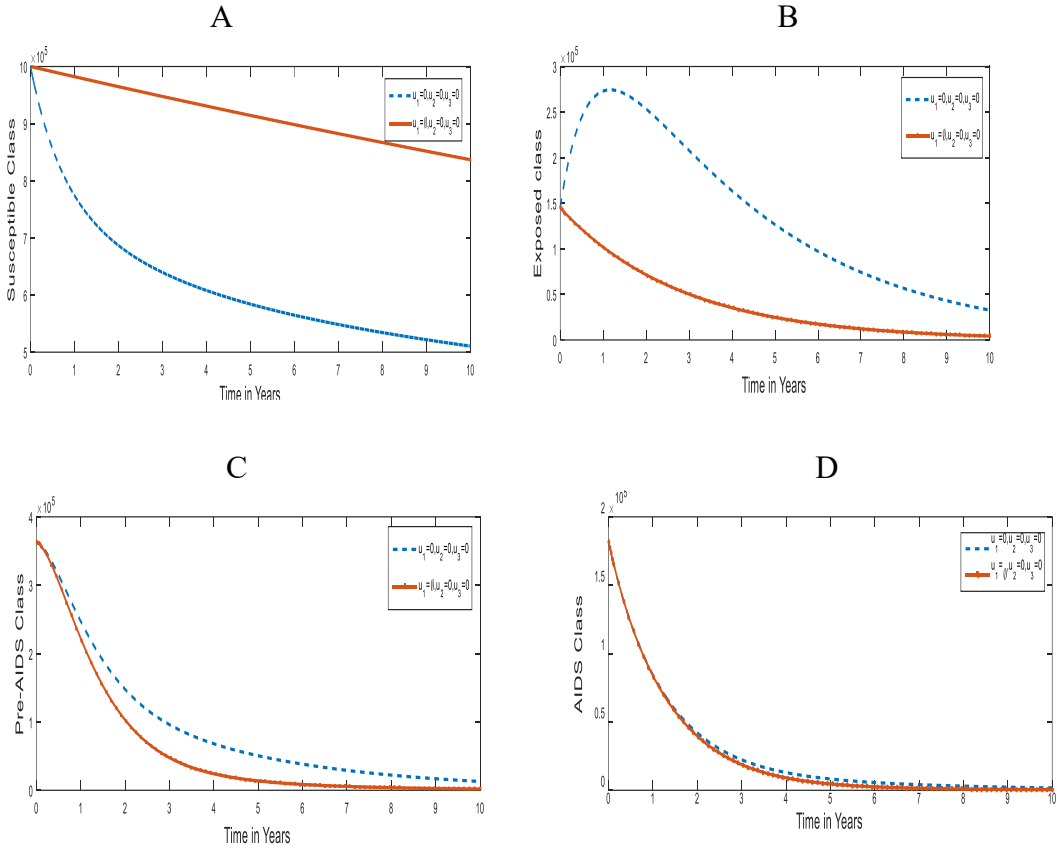

Fig.2 Simulations of the effect of Strategy 1 on the optimal control model: A. Susceptible population; B. under-3-day Exposed population; C. Pre-AIDS population; D. AIDS Population.

In Fig. 2A, there is a drastic increase in the population size of the susceptible class. This is because pre-contact prevention measures were aimed at protecting the population from initial contact with the virus, thereby reducing the population of the under-3-day Exposed and the Pre-AIDS population, as seen in Figures 2B and 2C. However, Figure 2D shows that the pre-contact prevention strategy alone has no appreciable effect on the individuals living with AIDS since they are already living with the virus, and there is no treatment. This intervention averted 2.4592 x 10^6 infected individuals. With this single control strategy, we notice that the

infected populations tend to zero from the 7th year of its optimal implementation, as shown in Figure 2C.

Strategy 2: The optimal use of post-contact preventive strategy (u_2) only on the transmission of HIV/AIDS in humans.

This involves the use of Post-Exposure Prophylaxis (PEP) intervention, which must be started within 72hours of exposure to HIV infection. It is aimed at preventing the under-3-day HIV exposed individuals from being infected.

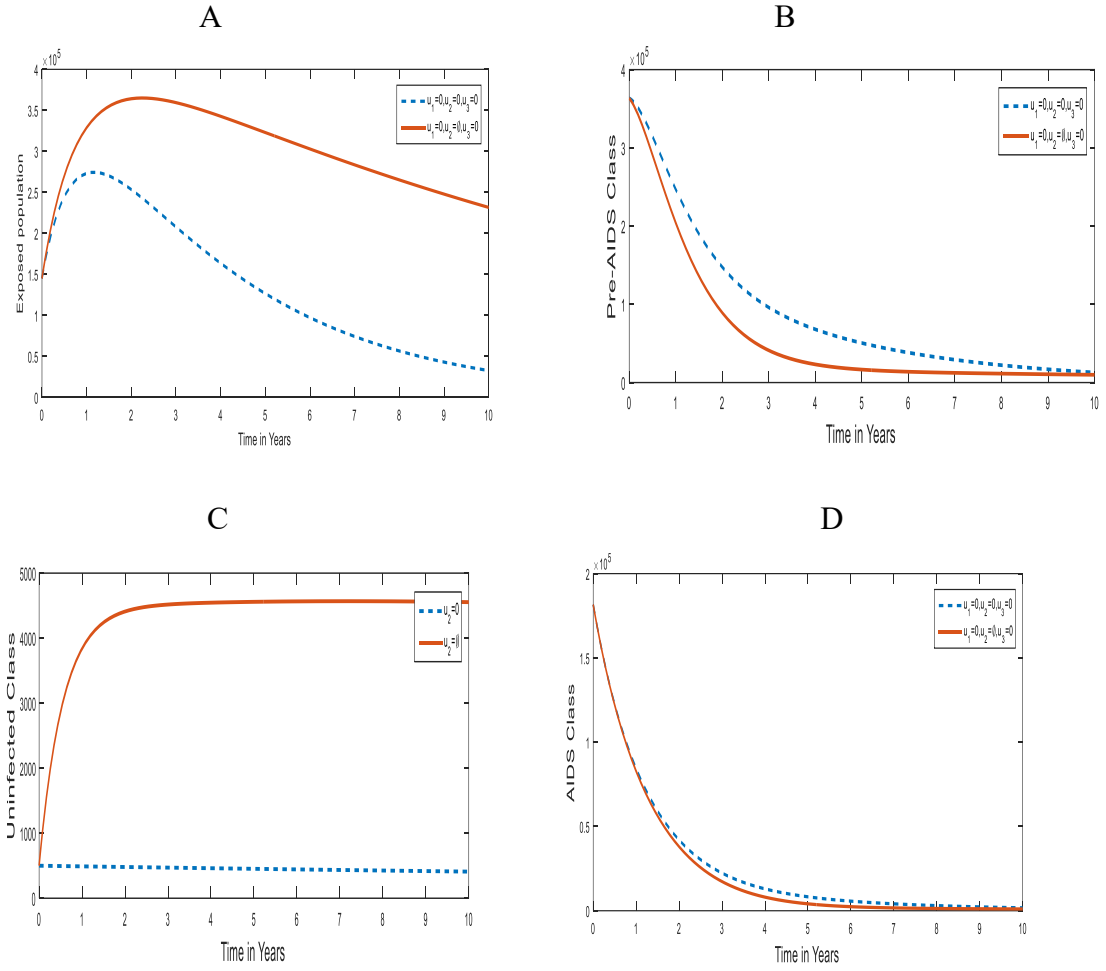

Fig.3 Simulations of the effect of Strategy 2 on the optimal control model: A. under-3-day Exposed population; B. Pre-AIDS population; C. Exposed-Uninfected population; D. AIDS Population.

Figure 2A shows a huge increase in the population of the under-3-day Exposed Class and a decrease in the number of Pre-AIDS individuals as depicted in Figure 2B. Also, on the application of this control strategy, many HIV Exposed individuals who could have been infected progress to the Exposed-uninfected Class. Fortunately, this is the only strategy that guarantees progression to the Exposed-uninfected class after an effective exposure to disease. This intervention also averted a total of 1.3518 x10^6 infected individuals.

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Strategy 3: The optimal control of HIV/AIDS transmission using Anti-Retroviral treatment (ART) (u_3) only on the infected humans.

This control measure is aimed at using anti-retroviral drugs to reduce the viral load in pre-AIDS and AIDS individuals. It is aimed at inhibiting or curtailing the transmission of the virus.

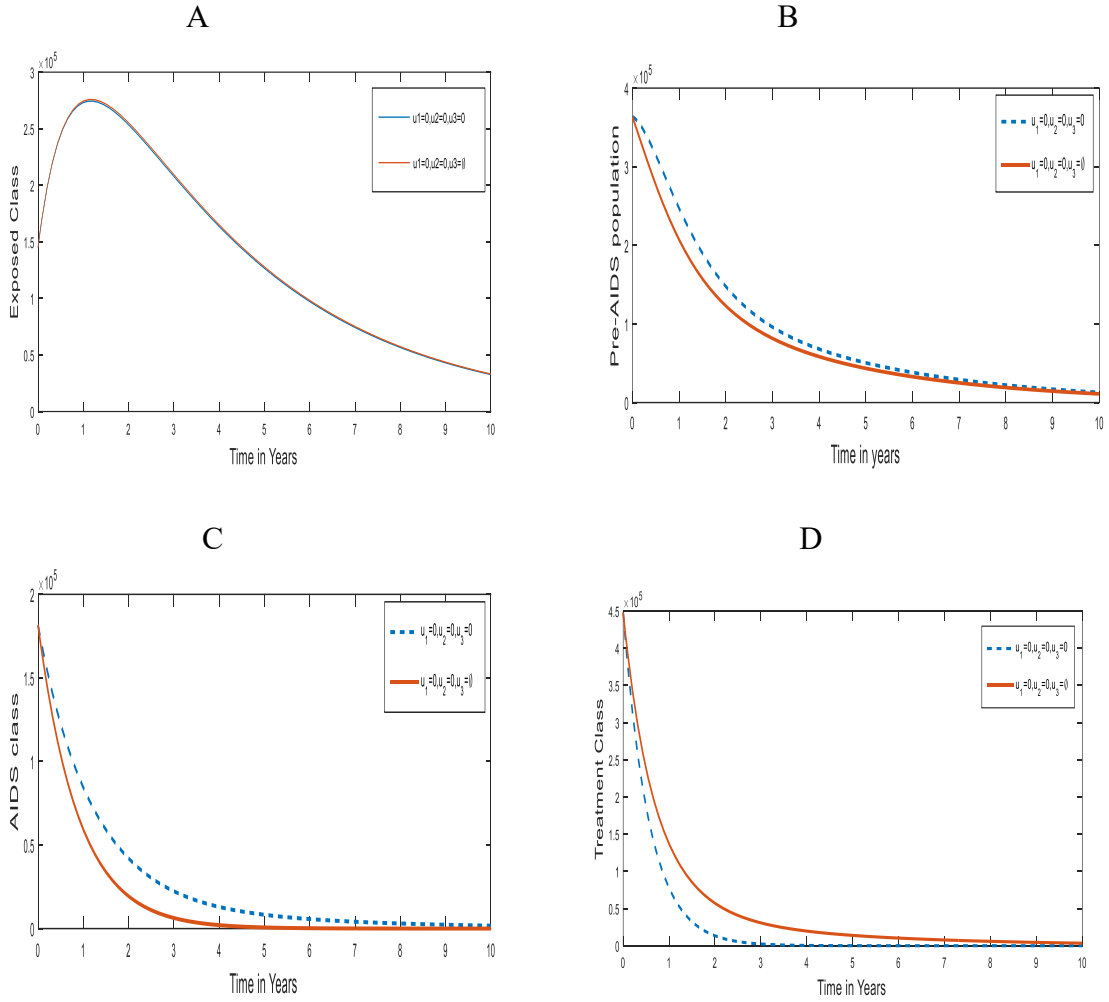

Fig.4 Simulations of the effect of Strategy 3 on the optimal control model: A. under-3-day Exposed population; B. pre-AIDS population; C. AIDS population; D. Treatment Population.

This control strategy does not have any effect on the exposed population, as noted in Fig.4A. The control measure has a minimal effect on the Pre-AIDS population, as shown in Fig. 4B. However, the control strategy averts a total of 0.4088 x10^6 infected individuals. The strategy is most useful in averting deaths due to the infection in the AIDS population. Anti- Retroviral treatments help to reduce the viral load in the AIDS population and progress to lower viral loads of the infected population. This is shown in Fig. 4C. The population of those receiving treatment rose at the implementation of this control measure.

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Strategy 4: The optimal use of combined control Strategies: pre-contact control (u_1) and post-contact control (u_2) strategies on the transmission of HIV/AIDS in humans.

Here, we investigate the impact of the combined control strategies of pre-contact and post-contact (use of PEP) on the optimal control of HIV/AIDS infection.

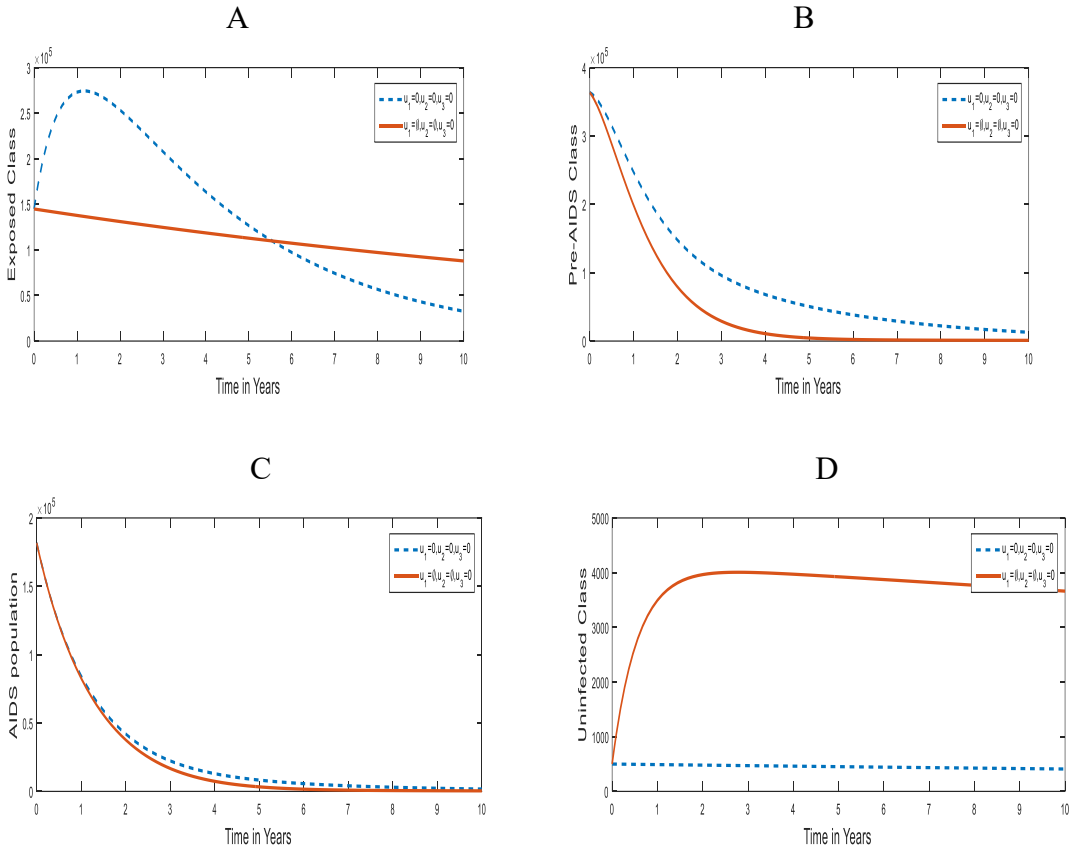

Fig.5 Simulations of the effect of Strategy 4 on the optimal control model: A. under-3-day Exposed population; B. pre-AIDS population; C. AIDS population; D. Uninfected Population.

Figure 5A shows a gradual reduction in the population of the exposed. This is due to the restrictions in the population of the under 3-day-exposed because of the use of PEP, as in Fig. 3A. This combined strategy promises the eradication of HIV/AIDS infection in about the 6th year of its application, as seen in Figures 5B and 5C. Figure 5D shows a high uninfected rate because of PEP intervention.

Strategy 5: The optimal use of combined control Strategies: pre-contact control (u_1) and Treatment with ART (u_3) strategies on the transmission of HIV/AIDS in humans.

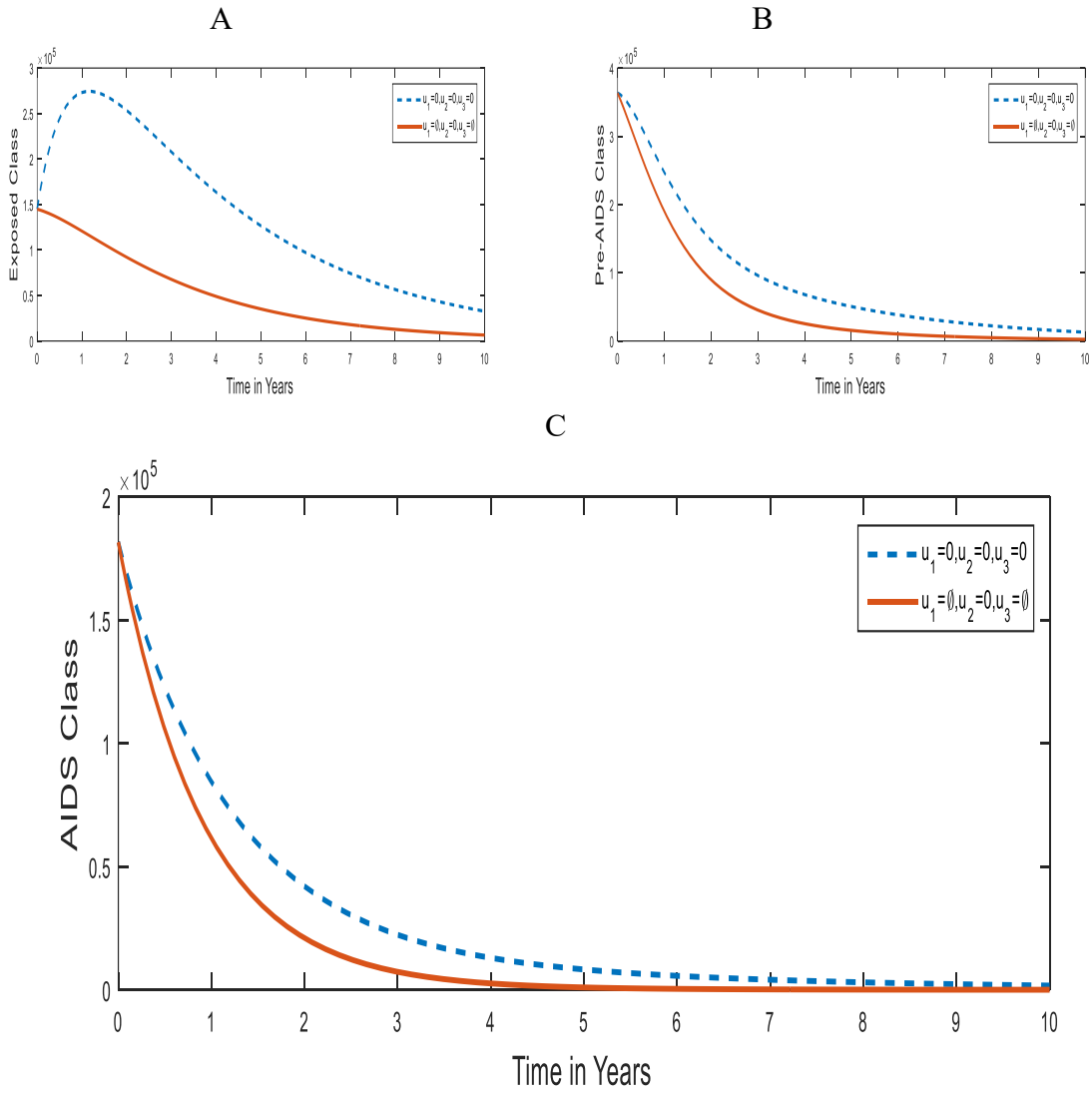

Fig.6 Simulations of the effect of Strategy 5 on the optimal control model: A. under-3-day Exposed population; B. Pre-AIDS population; C. AIDS population.

With this combined strategy, we notice a decline in the population of the under-3-day exposed the Pre-AIDS, and the AIDS classes. This is because both the HIV Pre-Contact and Treatment control measures put restrictions on them. However, no one will recover from the infection after an effective contact with the infected individual because of the absence of control strategy 2.

Strategy 6: The optimal use of combined control Strategies: post-contact control (u_2) and Treatment with ART control (u_3) strategies on the transmission of HIV/AIDS in humans.

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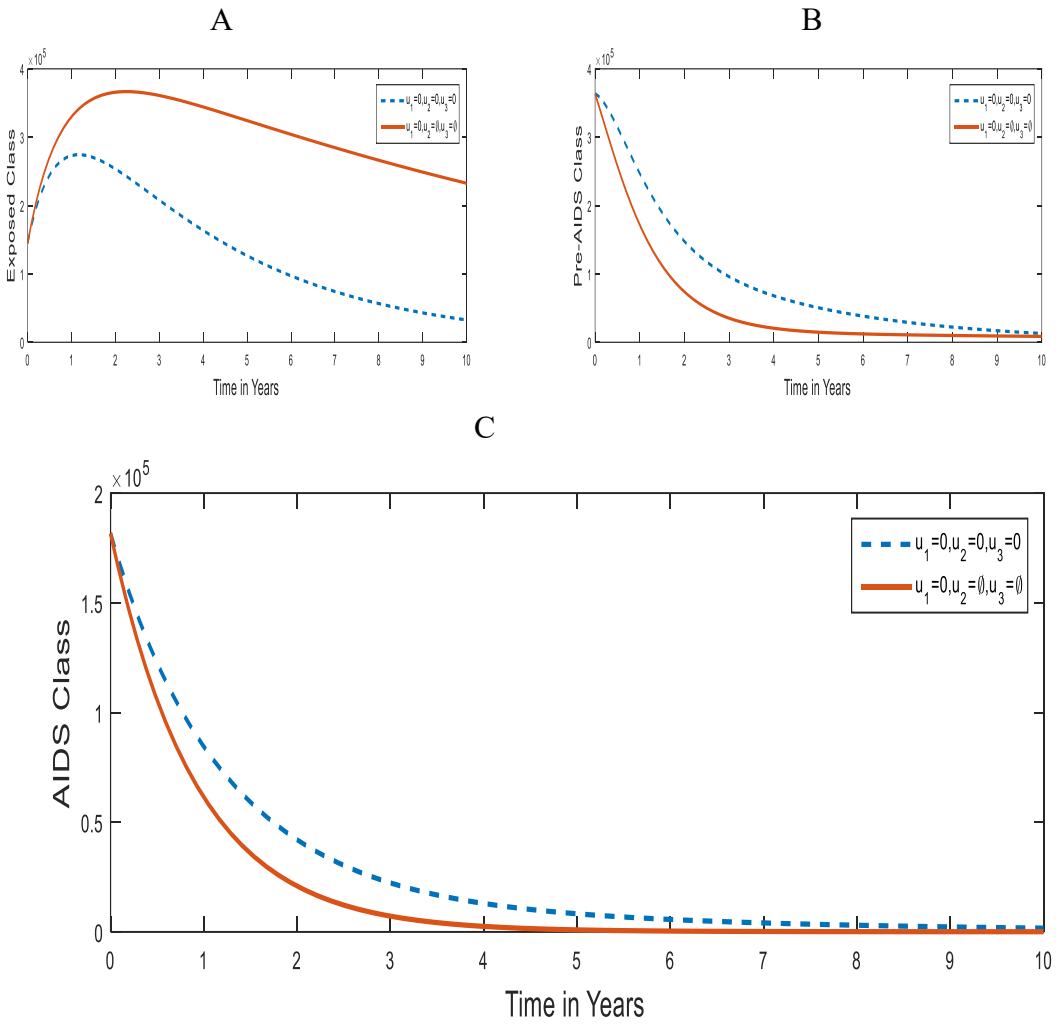

Fig.7 Simulations of the effect of Strategy 6 on the optimal control model: A. under-3-day Exposed population; B. pre-AIDS population; C. AIDS population.

The rise in the population of the Exposed Class is due to a lack of control measures for the susceptible class. Control with PEP intervention prevents the exposed population from contracting the infection, as shown in Figure 7A. The combined strategy also assures a low population of both the pre-AIDS and the AIDS classes but the infection will persist in the population, as shown in Figure 7B.

Strategy 7: The optimal use of the three control Strategies: Pre-contact control (u_1), post-contact control (u_2) and Treatment with ART control (u_3) strategies on the transmission of HIV/AIDS in humans.

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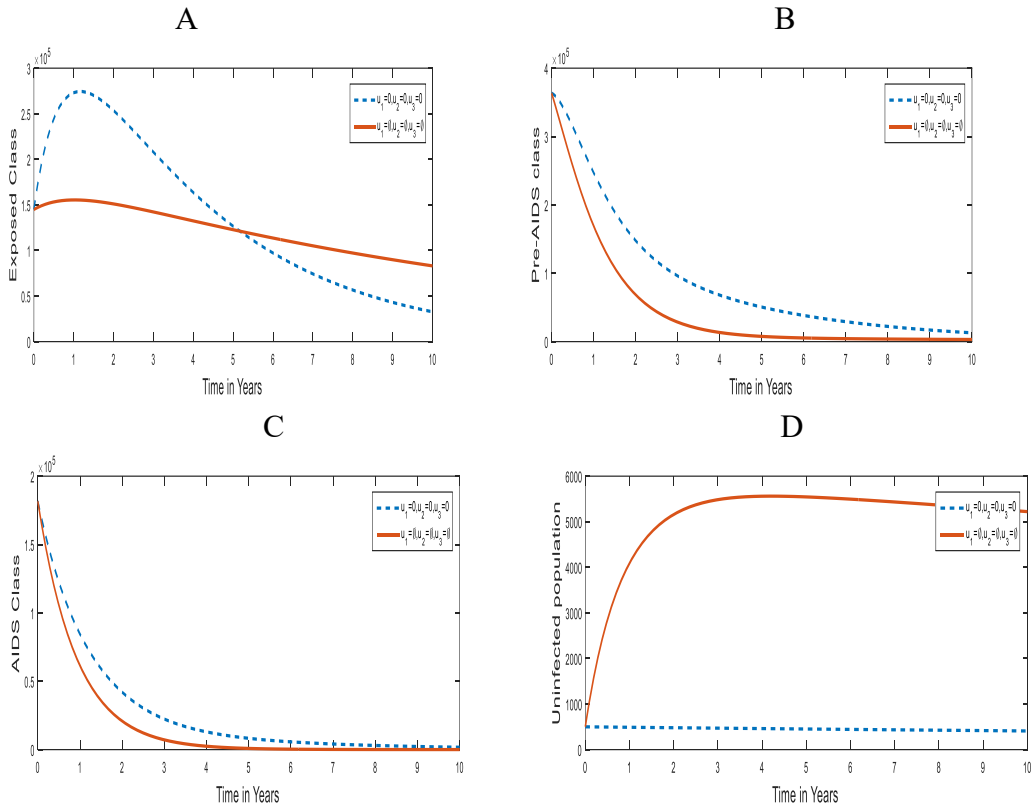

Fig.8 Simulations of the effect of Strategy 7 on the optimal control model: A. Exposed population; B. Pre-AIDS population; C. AIDS population; D. Exposed-Uninfected population.
Figure 8 shows the outcome of the three combined control strategies on transmission dynamics of HIV/AIDS infection. This outcome is similar to that obtained in strategy 4, such that each of them promises the eradication of the disease from the sixth year of implementation of the control measure. Both also promise a high rate of uninfected population even after an effective exposure to the disease.

Cost-effective analysis

Here, we search for the most cost-effective and accessible control strategies that can be employed to reduce or eradicate HIV/AIDS in a given population. We investigate and compare the cost of various control strategies to find the most cost-effective of the strategies. To derive this, we use the incremental cost-effectiveness ratio (CER), which compares the differences between the costs and health outcomes of two alternative intervention strategies that compete for the same resources (Cheneke K.R. 2023). It is the measure of the additional cost per additional health outcome.

$$ICER = \frac{Difference\ between\ costs}{Difference\ between\ health\ outcomes}$$

The number of infections averted is the difference between the number of individuals infected without the control strategy and the number infected with the control strategy. For

instance, the number of individuals infected without any control strategy is 9.4088 x 10^6, while those infected with strategy 1 control intervention is 6.9496 x 10^6. Therefore the intervention averted 2.4592 x 10^6 infections. Table 4 outlines the number of infections averted by each control measure, with the estimated costs.

Table 4: Total infection averted and the respective cost of implementing the strategies.

Control measures	Strategies	Total infection averted	Total cost	ICER
No Strategy	0	0	0	-
Single control	Strategy 1 Strategy 2 Strategy 3	2.4592 x 10^6 1.3518 x 10^6 0.4066 x 10^6	14980 12134 13576	0 0 0
Double control	Strategy 4 Strategy 5 Strategy 6	1.7255 x 10^6 3.8802 x 10^6 2.3696 x 10^6	46790 32680 43480	0 0 0
Triple control	Strategy 7	2.6958 x 10^6	82755	0

For the single control strategies:
We compare strategy 1 with the next less effective alternative (strategy 2) and compute as follows:

$$ICER(2) = \frac{12134}{1351800} = 0.00898$$

$$ICER(1) = \frac{14980 - 12134}{2459200 - 1351800} = 0.00257$$

Table 5: Comparison between strategies 2 and 1

Strategies	Total infection averted	Total cost	ICER
1 2	2.4592 x 10^6 1.3518 x 10^6	14980 12134	0.00257 0.00898

Since, $ICER(1) < ICER(2)$, we discard strategy 2 and construct a table and compare strategies 1and 3.

$$ICER(3) = \frac{13576}{408800} = 0.0332$$

$$ICER(1) = \frac{14980 - 13576}{2459200 - 408800} = 0.0006847$$

Table 6: ICER comparison of strategies 1 and 3

Strategies	Total infection averted	Total cost	ICER
1	2.4592×10^6	14980	0.0006847
3	0.4088×10^6	13576	0.0332

$ICER(1) < ICER(3)$, implying that strategy 3 is more expensive than strategy 1. Hence, strategy 1 is the least expensive of the three single strategies.

For double strategies:
Comparing strategies 4 and 5;

$$ICER(4) = \frac{46790}{1725500} = 0.02712$$

$$ICER(5) = \frac{32680 - 46790}{3880300 - 1725500} = -0.006548$$

Table 7: ICER comparison of strategies 4 and 5

Strategies	Total infection averted	Total cost	ICER
4	1.7255×10^6	46790	0.02712
5	3.8803×10^6	32680	-0.006548

$ICER(5) < ICER(4)$, we discard strategy 4 and compare strategies 5 and 6.

$$ICER(6) = \frac{43480}{2369600} = 0.01835$$

$$ICER(5) = \frac{32680 - 43480}{3880300 - 2369600} = -0.00715$$

Table 8: ICER comparison of strategies 5 and 6

Strategies	Total infection averted	Total cost	ICER
5	3.8803×10^6	32680	-0.00715
6	2.3696×10^6	43480	0.01835

$ICER(5) < ICER(6)$, we discard strategy 6 and conclude that strategy 5 is the least expensive among the double strategies.

The most cost-effective strategy of all the control strategies.

We have identified that among the single strategies, strategy 1 is the most cost-effective. In double strategies, control strategy 5 is the cheapest in implementation.

To determine the most cost-effective control measures, we compare and select among strategies 1, 5, and 7.

Comparing strategy 1and 5, we have:

$$ICER(1) = \frac{14980}{2459200} = 0.00609$$

$$ICER(5) = \frac{32680 - 14980}{3880300 - 2459200} = 0.0125$$

Table 9: ICER comparison of strategies 1 and 5

Strategies	Total infection averted	Total cost	ICER
1 5	2.4592 x 10^6 3.8803 x 10^6	14980 32680	0.00609 0.0125

$ICER(1) < ICER(5)$, we discard strategy 5 and compare strategies 1 and 7.

Comparing strategies 1 and 7, we have:

$$ICER(1) = \frac{14980}{2459200} = 0.00609$$

$$ICER(7) = \frac{82755 - 14680}{2695800 - 2459200} = 0.2877$$

Table 9: ICER comparison of strategies 1 and 7

Strategies	Total infection averted	Total cost	ICER
1 7	2.4592 x 10^6 2.6958 x 10^6	14980 82755	0.00609 0.2877

Table 9: ICER comparison of strategies 1 and 7

$ICER(1) < ICER(7)$, we discard strategy 7 and conclude that strategy 1 is the least expensive among all the strategies in implementation.

Hence, from the cost-effectiveness analysis, the Pre-contact control measure, which involves awareness campaigns, use of condoms, PrEP, and other pre-contact preventive interventions, is the best cost-effective strategy among all the discussed control strategies.

6. Conclusion and Recommendations.

Since there is no known cure for HIV/AIDS infection yet, it becomes pertinent that multiple control interventions should be employed to the disease. To further understand the spread and control of the disease, we developed a modified mathematical model of HIV/AIDS transmission dynamics with three main strategic controls. We first presented and analyzed the basic model without optimal control variables. The reproduction number (R_e) was obtained and used to determine the model's stability. It was found that the disease-free equilibrium would be asymptotically stable when $R_\varepsilon < 1$ and that the unique endemic equilibrium exists whenever $R_\varepsilon > 1$. We then extended the basic model by incorporating three time-dependent control variables to investigate the impact of various optimum control variables. The control strategies include: Pre-contact control strategies (u_1) , which involve all preventive measures aimed at avoiding the disease contact with the susceptible population; post-contact control (u_2) , which is the use of PEP on the exposed individuals to prevent the infection; and Treatment controls (u_3), used on the infected population to reduce the viral burden in them.

Pontryagin's maximal principle technique was used to obtain the optimal solutions. We used the fourth-order Runge-Kutta forward-backward sweep method to simulate the dynamics of the susceptible, Exposed, Pre-AIDS, Exposed but uninfected, and AIDS populations with and without these control measures. It was found that each control strategy reduced the HIV burden of the infected population, compared to that without control, as can be seen from Figure 2 through Figure 8. It can also be seen that strategy 7, which is the combination of all the single strategies, had the highest number of infected persons averted ($2.6958 \times 10^{6)}$. It is worth noting that only strategy 2, the use of PEP to control HIV infection, or its combinations with other strategies, can produce an exposed but uninfected population after an effective contact with the disease. However, the exposure must be under 3 days for it to work. Strategies 4 and 7 promise the eradication of the disease in the population within the sixth year, with a high uninfected population after exposure to the HIV infection, as shown in Figures 5 and 8. It was also deduced that strategy 1, the use of preventive measures to prevent susceptible populations' contact with the disease, is the most economical and effective among all the strategies.

It is therefore advised that, for minimal cost, respective Governments and health policy makers, especially in sub-Saharan Africa, should invest in this strategy 1. This could be achieved through public education and awareness campaigns on the prevention of HIV infection. Post-contact control strategy (use of PEP) might be costly to implement, under-3-day HIV exposed individuals are difficult to track, and the strategy is hard to execute within the required time limit; it should be aggressively pursued to prevent those accidentally exposed to the virus from contracting the infection. Hence, both PrEP and PEP drugs should be made available and affordable to the populace.

Data Availability

Data used to support the findings of this work are included in the manuscript.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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